4.3.14.1. *Stars of Filters on Boolean Lattices.* In this section we will consider the set of filters $\mathfrak{F}$ on a boolean lattice $\mathfrak{Z}$.

Note that $\mathfrak{P}$ is also a boolean lattice. We will take complements on $\mathfrak{P}$ without specifying that the complement is taken on $\mathfrak{P}$.

THEOREM 393. If $\mathfrak{Z}$ is a boolean lattice, $X \in \operatorname{up} \mathcal{A} \Leftrightarrow \overline{X} \notin \partial \mathcal{A}$ (where complement is taken on the boolean lattice $\mathfrak{P}$) for every $X \in \mathfrak{P}$, $\mathcal{A} \in \mathfrak{F}$.

PROOF. $X \in \operatorname{up} \mathcal{A} \Leftrightarrow X \sqsupseteq \mathcal{A} \Leftrightarrow \overline{X} \asymp^{\mathfrak{F}} \mathcal{A} \Leftrightarrow \overline{X} \notin \partial \mathcal{A}$ for any $X \in \mathfrak{P}$ (taking into account theorems 311, 379, 310). □

COROLLARY 394. If $\mathfrak{Z}$ is a boolean lattice and $\mathcal{A} \in \mathfrak{F}$ then

1°. $\partial \mathcal{A} = \left\{ \frac{\overline{X}}{X \in \mathfrak{P} \setminus \operatorname{up} \mathcal{A}} \right\}$;

2°. $\operatorname{up} \mathcal{A} = \left\{ \frac{\overline{X}}{X \in \mathfrak{P} \setminus \partial \mathcal{A}} \right\}$ (where complement is taken on the boolean lattice $\mathfrak{P}$).

COROLLARY 395. If $\mathfrak{Z}$ is a boolean lattice, ∂ is an injection.

For boolean lattices free stars bijectively correspond to filters:

THEOREM 396. FiXme: This theorem is an obvious consequence of isomorphisms between filters and free stars on the dual lattice. As such, this theorem should be removed from the text.If $\mathfrak{Z}$ is a boolean lattice, then for any set $S \in \mathscr{P}\mathfrak{P}$ there exists a filter $\mathcal{A}$ such that $\partial \mathcal{A} = S$ iff S is a free star.

PROOF. REMOVED THEOREM. □

PROPOSITION 397. If $\mathfrak{Z}$ is a boolean lattice then $\mathcal{A} \sqsubseteq \mathcal{B} \Leftrightarrow \partial \mathcal{A} \subseteq \partial \mathcal{B}$ for every $\mathcal{A}, \mathcal{B} \in \mathfrak{F}$.

PROOF.

$$\begin{aligned} &\partial \mathcal{A} \subseteq \partial \mathcal{B} \Leftrightarrow \\ \left\{ \frac{\overline{X}}{X \in \mathfrak{P} \setminus \mathcal{A}} \right\} \subseteq \left\{ \frac{\overline{X}}{X \in \mathfrak{P} \setminus \mathcal{B}} \right\} &\Leftrightarrow \\ \mathfrak{P} \setminus \mathcal{A} \subseteq \mathfrak{P} \setminus \mathcal{B} &\Leftrightarrow \\ \mathcal{A} \supseteq \mathcal{B} &\Leftrightarrow \\ &\mathcal{A} \sqsubseteq \mathcal{B}. \end{aligned}$$

□

COROLLARY 398. ∂ is a straight monotone map if $\mathfrak{Z}$ is a boolean lattice.FiXme: Remove it on behalf of more general fact that ∂ is an isomorphism?

THEOREM 399. If $\mathfrak{Z}$ is a boolean lattice then $\partial \bigsqcup^{\mathfrak{F}} S = \bigcup \langle \partial \rangle^* S$ for every $S \in \mathscr{P}\mathfrak{F}$.

PROOF. For boolean lattices ∂ is an order embedding from the poset $\mathfrak{F}$ to the complete lattice $\mathscr{P}\mathfrak{P}$. So accordingly the lemma 371 it is enough to prove that there exists $\mathcal{F} \in \mathfrak{F}$ such that $\partial \mathcal{F} = \bigcup \langle \partial \rangle^* S$. To prove this it is enough to show that $\bot^{\mathfrak{P}} \notin \bigcup \langle \partial \rangle^* S$ and

$$\forall A, B \in S : \left(A \sqcup^{\mathfrak{P}} B \in \bigcup \langle \partial \rangle^* S \Leftrightarrow A \in \bigcup \langle \partial \rangle^* S \vee B \in \bigcup \langle \partial \rangle^* S \right).$$

$\bot^{\mathfrak{P}} \notin \bigcup \langle \partial \rangle^* S$ is obvious.

Let $A \sqcup^{\mathfrak{P}} B \in \bigcup \langle \partial \rangle^* S$. Then there exists $Q \in \langle \partial \rangle^* S$ such that $A \sqcup^{\mathfrak{P}} B \in Q$. Then $A \in Q \vee B \in Q$, consequently $A \in \bigcup \langle \partial \rangle^* S \vee B \in \bigcup \langle \partial \rangle^* S$. Let now $A \in \bigcup \langle \partial \rangle^* S$. Then there exists $Q \in \langle \partial \rangle^* S$ such as $A \in Q$, consequently $A \sqcup^{\mathfrak{P}} B \in Q$ and $A \sqcup^{\mathfrak{P}} B \in \bigcup \langle \partial \rangle^* S$. □