Proof. Taking into account the corollary of the lemma, it is enough to prove that there exists $\mathcal{F} \in \mathfrak{F}$ such that $\mathcal{F} = \bigcap S$, that is that $R = \bigcap S$ is a filter.

R is nonempty because $\top \in R$. Let $A, B \in R$; then $\forall \mathcal{F} \in S : A, B \in \mathcal{F}$, consequently $\forall \mathcal{F} \in S : A \sqcap^{\mathfrak{Z}} B \in \mathcal{F}$. Consequently $A \sqcap^{\mathfrak{Z}} B \in \bigcap S = R$. So R is a filter base. Let $X \in R$ and $X \sqsubseteq Y \in \mathfrak{Z}$; then $\forall \mathcal{F} \in S : X \in \mathcal{F}$; $\forall \mathcal{F} \in S : Y \in \mathcal{F}$; $Y \in R$. So R is an upper set. □

Corollary 374. If $\mathfrak{Z}$ is a meet-semilattice with greatest element then $\mathfrak{F}$ is a complete lattice.

Corollary 375. If $\mathfrak{Z}$ is a meet-semilattice with greatest element then for any $\mathcal{A}, \mathcal{B} \in \mathfrak{F}$

$$\mathcal{A} \sqcup^{\mathfrak{F}} \mathcal{B} = \mathcal{A} \cap \mathcal{B}.$$

We will denote meets and joins on the lattice of filters just as $\sqcap$ and $\sqcup$.

Theorem 376. If $\mathfrak{Z}$ is a join-semilattice then $\mathfrak{F}$ is a join-semilattice and for any $\mathcal{A}, \mathcal{B} \in \mathfrak{F}$

$$\mathcal{A} \sqcup^{\mathfrak{F}} \mathcal{B} = \mathcal{A} \cap \mathcal{B}.$$

Proof. Taking into account the corollary of the lemma, it is enough to prove $R = \mathcal{A} \cap \mathcal{B}$ is a filter.

R is nonempty because there exists $X \in \mathcal{A}$ and $Y \in \mathcal{B}$ and $R \ni X \sqcup^{\mathfrak{Z}} Y$.

Let $A, B \in R$. Then $A, B \in \mathcal{A}$; so there exists $C \in \mathcal{A}$ such that $C \sqsubseteq A \wedge C \sqsubseteq B$. Analogously there exists $D \in \mathcal{B}$ such that $D \sqsubseteq A \wedge D \sqsubseteq B$. Let $E = C \sqcup^{\mathfrak{Z}} D$. Then $E \in \mathcal{A}$ and $E \in \mathcal{B}$; $E \in R$ and $E \sqsubseteq A \wedge E \sqsubseteq B$. So R is a filter base.

That R is an upper set is obvious. □

Theorem 377. If $\mathfrak{Z}$ is a distributive lattice then for $S \in \mathscr{P}\mathfrak{F} \setminus \{\emptyset\}$

$$\bigsqcap^{\mathfrak{F}} S = \left\{ \frac{K_0 \sqcap^{\mathfrak{Z}} \cdots \sqcap^{\mathfrak{Z}} K_n}{K_i \in \bigcup S \text{ where } i = 0, \ldots, n \text{ for } n \in \mathbb{N}} \right\}.$$

Proof. Let's denote the right part of the equality to be proven as R. First we will prove that R is a filter. R is nonempty because S is nonempty.

Let $A, B \in R$. Then $A = X_0 \sqcap^{\mathfrak{Z}} \cdots \sqcap^{\mathfrak{Z}} X_k$, $B = Y_0 \sqcap^{\mathfrak{Z}} \cdots \sqcap^{\mathfrak{Z}} Y_l$ where $X_i, Y_j \in \bigcup S$. So

$$A \sqcap^{\mathfrak{Z}} B = X_0 \sqcap^{\mathfrak{Z}} \cdots \sqcap^{\mathfrak{Z}} X_k \sqcap^{\mathfrak{Z}} Y_0 \sqcap^{\mathfrak{Z}} \cdots \sqcap^{\mathfrak{Z}} Y_l \in R.$$

Let filter $C \sqsupseteq A \in R$. Consequently (distributivity used)

$$C = C \sqcup^{\mathfrak{Z}} A = (C \sqcup^{\mathfrak{Z}} X_0) \sqcap^{\mathfrak{Z}} \cdots \sqcap^{\mathfrak{Z}} (C \sqcup^{\mathfrak{Z}} X_k).$$

$X_i \in P_i$ for some $P_i \in S$; $C \sqcup^{\mathfrak{Z}} X_i \in P_i$; $C \sqcup^{\mathfrak{Z}} X_i \in \bigcup S$; consequently $C \in R$.

We have proved that that R is a filter base and an upper set. So R is a filter.

Let $\mathcal{A} \in S$. Then $\mathcal{A} \subseteq \bigcup S$;

$$R \supseteq \bigsqcap^{\mathfrak{F}} S = \left\{ \frac{K_0 \sqcap^{\mathfrak{Z}} \cdots \sqcap^{\mathfrak{Z}} K_n}{K_i \in \mathcal{A} \text{ where } i = 0, \ldots, n \text{ for } n \in \mathbb{N}} \right\} = \mathcal{A}.$$

Consequently $\mathcal{A} \sqsupseteq R$.

Let now $\mathcal{B} \in \mathfrak{F}$ and $\forall \mathcal{A} \in S : \mathcal{A} \sqsupseteq \mathcal{B}$. Then $\forall \mathcal{A} \in S : \mathcal{A} \subseteq \mathcal{B}$; $\mathcal{B} \supseteq \bigcup S$. Thus $\mathcal{B} \supseteq T$ for every finite set $T \subseteq \bigcup S$. Consequently $\mathcal{B} \ni \bigsqcap^{\mathfrak{Z}} T$. Thus $\mathcal{B} \supseteq R$; $\mathcal{B} \sqsubseteq R$.

Comparing we get $\bigsqcap^{\mathfrak{F}} S = R$. □

Theorem 378. If $\mathfrak{Z}$ is a distributive lattice then for any $\mathcal{F}_0, \ldots, \mathcal{F}_m \in \mathfrak{F}$

$$\mathcal{F}_0 \sqcap^{\mathfrak{F}} \cdots \sqcap^{\mathfrak{F}} \mathcal{F}_m = \left\{ \frac{K_0 \sqcap^{\mathfrak{Z}} \cdots \sqcap^{\mathfrak{Z}} K_m}{K_i \in \mathcal{F}_i \text{ where } i = 0, \ldots, m} \right\}.$$