Proposition 351. If $\mathfrak{Z}$ is a meet-semilattice and S is a filter base on it, $A \in \mathfrak{Z}$, then $\langle A\sqcap\rangle^* S$ is also a filter base.

Proof. $\langle A\sqcap\rangle^* S \neq \emptyset$ because $S \neq \emptyset$.

Let $X, Y \in \langle A\sqcap\rangle^* S$. Then $X = A \sqcap X'$ and $Y = A \sqcap Y'$ where $X', Y' \in S$. There exists $Z' \in S$ such that $Z' \sqsubseteq X' \sqcap Y'$. So $X \sqcap Y = A \sqcap X' \sqcap Y' \sqsupseteq A \sqcap Z' \in \langle A\sqcap\rangle^* S$. □

4.3.3. Order of filters. Principal filters. I will make the set of filters $\mathfrak{F}$ into a poset by the order defined by the formula: $a \sqsubseteq b \Leftrightarrow a \supseteq b$.

Definition 352. The principal filter corresponding to an element $a \in \mathfrak{Z}$ is

$$\uparrow a = \left\{ \frac{x \in \mathfrak{Z}}{x \sqsupseteq a} \right\}.$$

Elements of $\mathfrak{P} = \langle\uparrow\rangle^* \mathfrak{Z}$ are called *principal filters*.

Obvious 353. Principal filters are filters.

Obvious 354. $\uparrow$ is an order embedding from $\mathfrak{Z}$ to $\mathfrak{F}$.

Corollary 355. $\uparrow$ is an order isomorphism between $\mathfrak{Z}$ and $\mathfrak{P}$.

Definition 356. For every poset $\mathfrak{Z}$ I call $(\mathfrak{F}; \mathfrak{P})$ the *primary filtrator* (for the base $\mathfrak{Z}$).

Proposition 357. $\uparrow K \sqsupseteq \mathcal{A} \Leftrightarrow K \in \mathcal{A}$.

Proof. $\uparrow K \sqsupseteq \mathcal{A} \Leftrightarrow \uparrow K \subseteq \mathcal{A} \Leftrightarrow K \in \mathcal{A}$. □

Proposition 358. $\operatorname{up} a = \langle\uparrow\rangle^* a$ for an element a of a primary filtrator.

Proof. For every $L \in \mathfrak{P}$ we have $L = \uparrow K$ for some $K \in \mathfrak{Z}$ and $L \in \operatorname{up} a \Leftrightarrow L \sqsupseteq a \Leftrightarrow \uparrow K \sqsupseteq a \Leftrightarrow K \in a \Leftrightarrow L \in \langle\uparrow\rangle^* a$. □

4.3.3.1. *Minimal and maximal filters.*

Obvious 359. The filter $\bot^{\mathfrak{F}} = \mathfrak{Z}$ (equal to the principal filter for the least element of $\mathfrak{Z}$ if it exists) is the least element of the poset of filters.

Proposition 360. If there exists greatest element $\top^{\mathfrak{Z}}$ of the poset $\mathfrak{Z}$ then $\top^{\mathfrak{F}} = \{1^{\mathfrak{Z}}\}$ is the greatest element of $\mathfrak{F}$.

Proof. Take into account that filters are nonempty. □

4.3.4. Primary filtrator is filtered. FiXme: Can the proof be simplified using the fact that "filtered" is the same as "semifiltered"?

Theorem 361. $\mathcal{A} = \bigsqcap^{\mathfrak{F}} \langle\uparrow\rangle^* \mathcal{A}$ for every filter $\mathcal{A}$ on a poset.

Proof. $\mathcal{A}$ is obviously a lower bound for $\langle\uparrow\rangle^* \mathcal{A}$. Let $\mathcal{B}$ be a lower bound for $\langle\uparrow\rangle^* \mathcal{A}$ that is

$$\forall K \in \mathcal{A} : \mathcal{B} \sqsubseteq \uparrow K$$

that is $\forall K \in \mathcal{A} : K \in \mathcal{B}$ that is $\mathcal{A} \subseteq \mathcal{B}$ that is $\mathcal{B} \sqsubseteq \mathcal{A}$. So $\mathcal{A}$ is the greatest lower bound for $\langle\uparrow\rangle^* \mathcal{A}$. □

Corollary 362. Every primary filtrator is filtered.

Corollary 363. Every primary filtrator is with join-closed core.

Proof. Theorem 292. □

Proposition 364. The filtrator $(\mathfrak{F}; \mathfrak{P})$ is with finitely meet-closed core if $\mathfrak{Z}$ is a meet-semilattice.

Proof. Theorem 311. □