4.3. Filters on a poset

4.3.1. Filters on posets. Let $\mathfrak{Z}$ be a poset.

DEFINITION 344. *Filter base* is a nonempty subset F of $\mathfrak{Z}$ such that

$$\forall X, Y \in F \exists Z \in F : (Z \sqsubseteq X \wedge Z \sqsubseteq Y).$$

OBVIOUS 345. A nonempty chain is a filter base.

DEFINITION 346. *Filter* is a subset of $\mathfrak{Z}$ which is both a filter base and an upper set.

I will denote the set of filters (for a given or implied poset $\mathfrak{Z}$) as $\mathfrak{F}$ and call $\mathfrak{F}$ the set of filters over the poset $\mathfrak{Z}$.

PROPOSITION 347. If $\top$ is the maximal element of $\mathfrak{Z}$ then $\top \in F$ for every filter F.

PROOF. If $\top \notin F$ then $\forall K \in \mathfrak{Z} : K \notin F$ and so F is empty what is impossible. □

PROPOSITION 348. Let S be a filter base on a poset. If $A_0, \ldots, A_n \in S$ ($n \in \mathbb{N}$), then

$$\exists C \in S : (C \sqsubseteq A_0 \wedge \ldots \wedge C \sqsubseteq A_n).$$

PROOF. It can be easily proved by induction. □

Dual of filters is called *ideals*. We do not use ideals in this work however. FiXme: Ideals will be used in a newer version of this text.

4.3.2. Filters on meet-semilattices.

THEOREM 349. If $\mathfrak{Z}$ is a meet-semilattice and F is a nonempty subset of $\mathfrak{Z}$ then the following conditions are equivalent:

1°. F is a filter.
2°. $\forall X, Y \in F : X \sqcap Y \in F$ and F is an upper set.
3°. $\forall X, Y \in \mathfrak{Z} : (X, Y \in F \Leftrightarrow X \sqcap Y \in F)$.

PROOF. □

1°⇒2°. Let F be a filter. Then F is an upper set. If $X, Y \in F$ then $Z \sqsubseteq X \wedge Z \sqsubseteq Y$ for some $Z \in F$. Because F is an upper set and $Z \sqsubseteq X \sqcap Y$ then $X \sqcap Y \in F$.

2°⇒1°. Let $\forall X, Y \in F : X \sqcap Y \in F$ and F be an upper set. We need to prove that F is a filter base. But it is obvious taking $Z = X \sqcap Y$ (we have also taken into account that $F \neq \emptyset$).

2°⇒3°. Let $\forall X, Y \in F : X \sqcap Y \in F$ and F be an upper set. Then

$$\forall X, Y \in \mathfrak{Z} : (X, Y \in F \Rightarrow X \sqcap Y \in F).$$

Let $X \sqcap Y \in F$; then $X, Y \in F$ because F is an upper set.

3°⇒2°. Let

$$\forall X, Y \in \mathfrak{Z} : (X, Y \in F \Leftrightarrow X \sqcap Y \in F).$$

Then $\forall X, Y \in F : X \sqcap Y \in F$. Let $X \in F$ and $X \sqsubseteq Y \in \mathfrak{Z}$. Then $X \sqcap Y = X \in F$. Consequently $X, Y \in F$. So F is an upper set.

PROPOSITION 350. Let S be a filter base on a meet-semilattice. If $A_0, \ldots, A_n \in S$ ($n \in \mathbb{N}$), then

$$\exists C \in S : C \sqsubseteq A_0 \sqcap \cdots \sqcap A_n.$$

PROOF. It can be easily proved by induction. □