PROOF. Let a be an atom of the lattice $\mathfrak{A}$. We have for every $X, Y \in \mathfrak{Z}$

$$\begin{aligned} X \sqcup^{\mathfrak{Z}} Y \in \operatorname{up} a \Leftrightarrow \\ X \sqcup^{\mathfrak{A}} Y \in \operatorname{up} a \Leftrightarrow \\ X \sqcup^{\mathfrak{A}} Y \sqsupseteq a \Leftrightarrow \\ X \sqcup^{\mathfrak{A}} Y \not\asymp^{\mathfrak{A}} a \Leftrightarrow \\ X \not\asymp^{\mathfrak{A}} a \vee Y \not\asymp^{\mathfrak{A}} a \Leftrightarrow \\ X \sqsupseteq a \vee Y \sqsupseteq a \Leftrightarrow \\ X \in \operatorname{up} a \vee Y \in \operatorname{up} a. \end{aligned}$$

□

4.2.8. Some Criteria.

THEOREM 320. For a semifiltered, star-separable, down-aligned filtrator $(\mathfrak{A}; \mathfrak{Z})$ with finitely meet closed and separable core where $\mathfrak{Z}$ is a complete boolean lattice and both $\mathfrak{Z}$ and $\mathfrak{A}$ are atomistic lattices the following conditions are equivalent for any $\mathcal{F} \in \mathfrak{A}$:

1°. $\mathcal{F} \in \mathfrak{Z}$;
2°. $\forall S \in \mathscr{P}\mathfrak{A} : \left(\mathcal{F} \sqcap^{\mathfrak{A}} \bigsqcup^{\mathfrak{A}} S \neq \bot \Rightarrow \exists \mathcal{K} \in S : \mathcal{F} \sqcap^{\mathfrak{A}} \mathcal{K} \neq \bot\right)$;
3°. $\forall S \in \mathscr{P}\mathfrak{Z} : \left(\mathcal{F} \sqcap^{\mathfrak{A}} \bigsqcup^{\mathfrak{A}} S \neq \bot \Rightarrow \exists K \in S : \mathcal{F} \sqcap^{\mathfrak{A}} K \neq \bot\right)$.

PROOF. Our filtrator is with join-closed core (theorem 292).

1°⇒2°. Let $\mathcal{F} \in \mathfrak{Z}$. Then (taking into account the proposition 310)

$$\mathcal{F} \sqcap^{\mathfrak{A}} \bigsqcup^{\mathfrak{A}} S \neq \bot \Leftrightarrow \overline{\mathcal{F}} \not\sqsupseteq \bigsqcup^{\mathfrak{A}} S \Rightarrow \exists \mathcal{K} \in S : \overline{\mathcal{F}} \not\sqsupseteq \mathcal{K} \Leftrightarrow \exists \mathcal{K} \in S : \mathcal{F} \sqcap^{\mathfrak{A}} \mathcal{K} \neq \bot.$$

2°⇒3°. Obvious.

3°⇒1°. Let the formula (3) be true. Then for $L \in \mathfrak{Z}$ and $S = \operatorname{atoms}^{\mathfrak{Z}} L$ it takes the form

$$\mathcal{F} \sqcap^{\mathfrak{A}} \bigsqcup^{\mathfrak{A}} \neq \bot \Rightarrow \exists K \in S : \mathcal{F} \sqcap^{\mathfrak{A}} K \neq \bot$$

that is $\mathcal{F} \sqcap^{\mathfrak{A}} L \neq \bot \Rightarrow \exists K \in S : \mathcal{F} \sqcap^{\mathfrak{A}} K \neq \bot$ because $\bigsqcup^{\mathfrak{A}} \operatorname{atoms}^{\mathfrak{Z}} L = \bigsqcup^{\mathfrak{Z}} \operatorname{atoms}^{\mathfrak{Z}} L = L$. That is $\mathcal{F} \sqcap^{\mathfrak{A}} L \Rightarrow \mathcal{F} \sqcap^{\mathfrak{A}} K_L \neq \bot$ where $K_L \in S$. Thus K_L is an atom of both $\mathfrak{A}$ and $\mathfrak{Z}$ (see the theorem 316), so having $\mathcal{F} \sqcap^{\mathfrak{A}} L \neq \bot \Rightarrow \mathcal{F} \sqsupseteq K_L$. Let

$$F = \bigsqcup^{\mathfrak{Z}} \left\{ \frac{K_L}{L \in \mathfrak{Z}, \mathcal{F} \sqcap^{\mathfrak{A}} L \neq \bot} \right\}.$$

Then

$$F = \bigsqcup^{\mathfrak{A}} \left\{ \frac{K_L}{L \in \mathfrak{Z}, \mathcal{F} \sqcap^{\mathfrak{A}} L \neq \bot} \right\}.$$

Obviously $F \sqsubseteq \mathcal{F}$. We have $L \sqcap^{\mathfrak{A}} \mathcal{F} \neq \bot \Rightarrow K_L \sqcap^{\mathfrak{Z}} F \neq \bot \Rightarrow L \sqcap^{\mathfrak{Z}} F \neq \bot \Rightarrow L \sqcap^{\mathfrak{A}} F \neq \bot$, thus by star separability of our filtrator $\mathcal{F} \sqsubseteq F$ and so $\mathcal{F} = F \in \mathfrak{Z}$.

□

DEFINITION 321. Let S be a subset of a meet-semilattice. The *filter base generated by* S is the set

$$[S]_{\sqcap} = \left\{ \frac{a_0 \sqcap \dots \sqcap a_n}{a_i \in S, i = 0, 1, \dots} \right\}.$$