PROOF. We will prove only the first as the second is dual.

$$\begin{aligned} B \asymp^{\mathfrak{A}} \mathcal{A} &\Leftrightarrow \\ \exists A \in \operatorname{up}\mathcal{A} : B \asymp^{\mathfrak{A}} A &\Leftrightarrow \\ \exists A \in \operatorname{up}\mathcal{A} : B \sqcap^{\mathfrak{A}} A = \bot &\Leftrightarrow \\ \exists A \in \operatorname{up}\mathcal{A} : B \sqcap^{\mathfrak{Z}} A = \bot &\Leftrightarrow \\ \exists A \in \operatorname{up}\mathcal{A} : \overline{B} \sqsupseteq A &\Leftrightarrow \\ \overline{B} \in \operatorname{up}\mathcal{A} &\Leftrightarrow \\ \overline{B} \sqsupseteq \mathcal{A}. \end{aligned}$$

□

4.2.4. Characterization of Finitely Meet-Closed Filtrators.

THEOREM 311. The following are equivalent for a filtrator $(\mathfrak{A}; \mathfrak{Z})$ whose core is a meet semilattice such that $\forall a \in \mathfrak{A} : \operatorname{up} a \neq \emptyset$:

1°. The filtrator is finitely meet-closed.
2°. $\operatorname{up} a$ is a filter for every $a \in \mathfrak{A}$.

PROOF.

1°⇒2°. Let $X, Y \in \operatorname{up} a$. Then $X \sqcap^{\mathfrak{Z}} Y = X \sqcap^{\mathfrak{A}} Y \sqsupseteq a$. That $\operatorname{up} a$ is an upper set is obvious. So taking into account that $\operatorname{up} a \neq \emptyset$, $\operatorname{up} a$ is a filter.

2°⇒1°. It is enough to prove that $a \sqsubseteq A, B \Rightarrow a \sqsubseteq A \sqcap^{\mathfrak{Z}} B$ for every $A, B \in \mathfrak{A}$. Really:

$$a \sqsubseteq A, B \Rightarrow A, B \in \operatorname{up} a \Rightarrow A \sqcap^{\mathfrak{Z}} B \in \operatorname{up} a \Rightarrow a \sqsubseteq A \sqcap^{\mathfrak{Z}} B.$$

□

4.2.5. Stars of Elements of Filtrators.

DEFINITION 312. Let $(\mathfrak{A}; \mathfrak{Z})$ be a filtrator. *Core star* of an element a of a filtrator is

$$\partial a = \left\{ \frac{x \in \mathfrak{Z}}{x \not\asymp^{\mathfrak{A}} a} \right\}.$$

PROPOSITION 313. $\operatorname{up} a \subseteq \partial a$ for any non-least element a of a filtrator.

PROOF. For any element $X \in \mathfrak{Z}$

$$X \in \operatorname{up} a \Rightarrow a \sqsubseteq X \wedge a \sqsubseteq a \Rightarrow X \not\asymp^{\mathfrak{A}} a \Rightarrow X \in \partial a.$$

□

THEOREM 314. Let $(\mathfrak{A}; \mathfrak{Z})$ be a distributive lattice filtrator with least element and finitely join-closed core which is a join semilattice. Then ∂a is a free star for each $a \in \mathfrak{A}$.

PROOF. For every $A, B \in \mathfrak{Z}$

$$\begin{aligned} A \sqcup^{\mathfrak{Z}} B \in \partial a &\Leftrightarrow \\ A \sqcup^{\mathfrak{A}} B \in \partial a &\Leftrightarrow \\ (A \sqcup^{\mathfrak{A}} B) \sqcap^{\mathfrak{A}} a \neq \bot^{\mathfrak{A}} &\Leftrightarrow \\ (A \sqcap^{\mathfrak{A}} a) \sqcup^{\mathfrak{A}} (B \sqcap^{\mathfrak{A}} a) \neq \bot^{\mathfrak{A}} &\Leftrightarrow \\ A \sqcap^{\mathfrak{A}} a \neq \bot^{\mathfrak{A}} \vee B \sqcap^{\mathfrak{A}} a \neq \bot^{\mathfrak{A}} &\Leftrightarrow \\ A \in \partial a \vee B \in \partial a. \end{aligned}$$

That ∂a doesn't contain $\bot^{\mathfrak{A}}$ is obvious. □