CHAPTER 4

Filters and filtrators

This chapter is based on my article [29].

This chapter is grouped in the following way:

- First it goes a short introduction in pedagogical order (first less general stuff and examples, last the most general stuff):
  - filters on a set;
  - filters on a meet-semilattice;
  - filters on a poset;
  - filtrators.
- Then it goes the formal part in the order from the most general to the least general:
  - filtrators;
  - filters on a poset;
  - filters on a set.

FiXme: Rewrite this paragraph accordingly the rewrite plan.Most theorems about filtrators (and also some theorems about filters on posets) have the form $A \Rightarrow B$ where A is the specific theorem condition and B is the main theorem statement. To most such theorems correspond simple B when we restrict to consideration only to the filtrator of filters on a fixed set. In some sense only B here is important, A here is a technical condition. So reading theorems about filtrators concentrate on the theorem statement rather than on theorem conditions.

4.1. Introduction to filters and filtrators

4.1.1. Filters on a set. We sometimes want to define something resembling an infinitely small (or infinitely big) set, for example the infinitely small interval near 0 on the real line. Of course there is no such set, just like as there is no natural number which is the difference $2 - 3$. To overcome this shortcoming we introduce whole numbers, and $2-3$ becomes well defined. In the same way to consider things which are like infinitely small (or infinitely big) sets we introduce *filters*.

An example of a filter is the infinitely small interval near 0 on the real line. To come to infinitely small, we consider all intervals $(-\epsilon; \epsilon)$ for all $\epsilon > 0$. This filter consists of all intervals $(-\epsilon; \epsilon)$ for all $\epsilon > 0$ and also all subsets of $\mathbb{R}$ containing such intervals as subsets. Informally speaking, this is the greatest filter contained in every interval $(-\epsilon; \epsilon)$ for all $\epsilon > 0$.

Definition 267. A filter on a set $\mho$ is a $\mathcal{F} \in \mathscr{P}\mathscr{P}\mho$ such that:

1°. $\forall A, B \in \mathcal{F} : A \cap B \in \mathcal{F}$;

2°. $\forall A, B \in \mathscr{P}\mho : (A \in \mathcal{F} \land B \supseteq A \Rightarrow B \in \mathcal{F})$.

Exercise 268. Verify that the above introduced infinitely small interval near 0 on the real line is a filter on $\mathbb{R}$.

Exercise 269. Describe “the neighborhood of positive infinity” filter on $\mathbb{R}$.

Definition 270. A filter not containing empty set is called a *proper filter*.