3.8.4.7. *Associativity of ordinated product.* Let f be an ordinal variadic function.

Let S be an ordinal indexed family of functions of ordinal indexed families of functions each taking an ordinal number of arguments in a set X.

I call f *infinite associative* when

1°. $f(f\circ S) = f(\operatorname{concat} S)$ for every S;
2°. $f(\llbracket x\rrbracket) = x$ for $x\in X$.

Infinite associativity implies associativity.

Proposition 260. Let f be an infinitely associative function taking an ordinal number of arguments in a set X. Define $x\star y = f\llbracket x;y\rrbracket$ for $x,y\in X$. Then the binary operation $\star$ is associative.

Proof. Let $x,y,z\in X$. Then $(x\star y)\star z = f\llbracket f\llbracket x;y\rrbracket;z\rrbracket = f(f\llbracket x;y\rrbracket;f\llbracket z\rrbracket) = f\llbracket x;y;z\rrbracket$. Similarly $x\star(y\star z) = f\llbracket x;y;z\rrbracket$. So $(x\star y)\star z = x\star(y\star z)$. □

Concatenation is associative. First we will prove some lemmas.

Let a and b be functions on a poset. Let $a\sim b$ iff there exist an order isomorphism f such that $a = b\circ f$. Evidently $\sim$ is an equivalence relation.

Obvious 261. $\operatorname{concat} a = \operatorname{concat} b \Leftrightarrow \operatorname{uncurry}(a)\sim\operatorname{uncurry}(b)$ for every ordinal indexed families a and b of functions taking an ordinal number of arguments.

Thank to the above, we can reduce properties of concat to properties of uncurry.

Lemma 262. $a\sim b\Rightarrow \operatorname{uncurry} a\sim\operatorname{uncurry} b$ for every ordinal indexed families a and b of functions taking an ordinal number of arguments.

Proof. There exist an order isomorphism f such that $a = b\circ f$.

$\operatorname{uncurry}(a)(x;y) = (ax)y = (bfx)y = \operatorname{uncurry}(b)(fx;y) = \operatorname{uncurry}(b)g(x;y)$ where $g(x;y) = (fx;y)$.

g is an order isomorphism because $g(x_0;y_0)\geq g(x_1;y_1)\Leftrightarrow (x_0;y_0)\geq(x_1;y_1)$. (Injectivity and surjectivity are obvious.) □

Lemma 263. Let $a_i\sim b_i$ for every i. Then $\operatorname{uncurry} a\sim\operatorname{uncurry} b$ for every ordinal indexed families a and b of ordinal indexed families of functions taking an ordinal number of arguments.

Proof. Let $a_i = b_i\circ f_i$ where f_i is an order isomorphism for every i.

$\operatorname{uncurry}(a)(i;y) = a_i y = b_i f_i y = \operatorname{uncurry}(b)(i;f_i y) = \operatorname{uncurry}(b)g(i;y) = (\operatorname{uncurry}(b)\circ g)(i;y)$ where $g(i;y) = (i;f_i y)$.

g is an order isomorphism because $g(i;y_0)\geq g(i;y_1)\Leftrightarrow f_i y_0\geq f_i y_1\Leftrightarrow y_0\geq y_1$ and $i_0>i_1\Rightarrow g(i;y_0)>g(i;y_1)$. (Injectivity and surjectivity are obvious.) □

Let now S be an ordinal indexed family of ordinal indexed families of functions taking an ordinal number of arguments.

Lemma 264. $\operatorname{uncurry}(\operatorname{uncurry}\circ S)\sim\operatorname{uncurry}(\operatorname{uncurry} S)$.

Proof. $\operatorname{uncurry}\circ S = \lambda i\in S : \operatorname{uncurry}(S_i)$;

$(\operatorname{uncurry}(\operatorname{uncurry}\circ S))((i;x);y) = (\operatorname{uncurry} S_i)(x;y) = (S_i x)y$;

$(\operatorname{uncurry}(\operatorname{uncurry} S))((i;x);y) = ((\operatorname{uncurry} S)(i;x))y = (S_i x)y$.

Thus $(\operatorname{uncurry}(\operatorname{uncurry}\circ S))((i;x);y) = (\operatorname{uncurry}(\operatorname{uncurry} S))((i;x);y)$ and thus evidently $\operatorname{uncurry}(\operatorname{uncurry}\circ S)\sim\operatorname{uncurry}(\operatorname{uncurry} S)$. □

Theorem 265. concat is an infinitely associative function.