3.8.4.5. *Definition with composition for every multiplier.* $q(F)_i \stackrel{\text{def}}{=} (\operatorname{curry}(\bigoplus(\operatorname{arity} \circ F)))i$.

PROPOSITION 246. $\prod^{(\mathrm{ord})} F = \left\{ \frac{L \in \mho^{\sum(\operatorname{arity} \circ F)}}{\forall i \in \operatorname{dom} F : L \circ q(F)_i \in \operatorname{GR} F_i} \right\}$.

PROOF. $\operatorname{GR} \prod^{(\mathrm{ord})} F = \left\{ \frac{\operatorname{concat} z}{z \in \prod(\operatorname{GR} \circ F)} \right\}$;
$\operatorname{GR} \prod^{(\mathrm{ord})} F = \left\{ \frac{\operatorname{uncurry}(z) \circ \left(\bigoplus(\operatorname{arity} \circ f)\right)^{-1}}{z \in \prod_{i \in \operatorname{dom} F} \mho^{\operatorname{arity} F_i}, \forall i \in \operatorname{dom} F : z(i) \in \operatorname{GR} F_i} \right\}$.
Let $L = \operatorname{uncurry}(z)$. Then $z = \operatorname{curry}(L)$.
$\operatorname{GR} \prod^{(\mathrm{ord})} F = \left\{ \frac{L \circ \left(\bigoplus(\operatorname{arity} \circ f)\right)^{-1}}{\operatorname{curry}(L) \in \prod_{i \in \operatorname{dom} F} \mho^{\operatorname{arity} F_i}, \forall i \in \operatorname{dom} F : \operatorname{curry}(L) i \in \operatorname{GR} F_i} \right\}$;
$\operatorname{GR} \prod^{(\mathrm{ord})} F = \left\{ \frac{L \circ \left(\bigoplus(\operatorname{arity} \circ f)\right)^{-1}}{L \in \mho^{\coprod_{i \in \operatorname{dom} F} \operatorname{arity} F_i}, \forall i \in \operatorname{dom} F : \operatorname{curry}(L) i \in \operatorname{GR} F_i} \right\}$;
$\operatorname{GR} \prod^{(\mathrm{ord})} F = \left\{ \frac{L \in \mho^{\sum(\operatorname{arity} \circ f)}}{\forall i \in \operatorname{dom} F : \operatorname{curry}\left(L \circ \bigoplus(\operatorname{arity} \circ F)\right) i \in \operatorname{GR} F_i} \right\}$;
$(\operatorname{curry}(L \circ \bigoplus(\operatorname{arity} \circ F))i)x = L((\operatorname{curry}(\bigoplus(\operatorname{arity} \circ F))i)x) = L(q(F)_i x) = (L \circ q(F)_i)x$;
$\operatorname{curry}(L \circ \bigoplus(\operatorname{arity} \circ F))i = L \circ q(F)_i$;
$\prod^{(\mathrm{ord})} F = \left\{ \frac{L \in \mho^{\sum(\operatorname{arity} \circ F)}}{\forall i \in \operatorname{dom} F : L \circ q(F)_i \in \operatorname{GR} F_i} \right\}$. □

COROLLARY 247. $\prod^{(\mathrm{ord})} F = \left\{ \frac{L \in \left(\bigcup \operatorname{im}(\operatorname{GR} \circ F)\right)^{\sum(\operatorname{arity} \circ F)}}{\forall i \in \operatorname{dom} F : L \circ q(F)_i \in \operatorname{GR} F_i} \right\}$.

COROLLARY 248. $\prod^{(\mathrm{ord})} F$ is small if F is small.

3.8.4.6. *Definition with shifting arguments.* Let $F'_i = \left\{ \frac{L \circ \Pr_1 |_{\{i\} \times \operatorname{arity} F_i}}{L \in \operatorname{GR} F_i} \right\}$.

PROPOSITION 249. $F'_i = \left\{ \frac{L \circ \Pr_1 |_{\{i\} \times \mho}}{L \in \operatorname{GR} F_i} \right\}$.

PROOF. If $L \in \operatorname{GR} F_i$ then $\operatorname{dom} L = \operatorname{arity} F_i$. Thus

$$L \circ \Pr_1 |_{\{i\} \times \operatorname{arity} F_i} = L \circ \Pr_1 |_{\{i\} \times \operatorname{dom} L} = L \circ \Pr_1 |_{\{i\} \times \mho}.$$

□

PROPOSITION 250. F'_i is an $(\{i\} \times \operatorname{arity} F_i)$-ary relation.

PROOF. We need to prove that $\operatorname{dom}\left(L \circ \Pr_1 |_{\{i\} \times \operatorname{arity} F_i}\right) = \{i\} \times \operatorname{arity} F_i$ for $L \in \operatorname{GR} F_i$, but that's obvious. □

OBVIOUS 251. $\coprod(\operatorname{arity} \circ F) = \bigcup_{i \in \operatorname{dom} F} (\{i\} \times \operatorname{arity} F_i) = \bigcup_{i \in \operatorname{dom} F} \operatorname{dom} F'_i$.

LEMMA 252. $P \in \prod_{i \in \operatorname{dom} F} F'_i \Leftrightarrow \operatorname{curry}(\bigcup \operatorname{im} P) \in \prod(\operatorname{GR} \circ F)$ for a $\operatorname{dom} F$ indexed family P where $P_i \in \mho^{\{i\} \times \operatorname{arity} F_i}$ for every $i \in \operatorname{dom} F$, that is for $P \in \coprod_{i \in \operatorname{dom} F} \mho^{\{i\} \times \operatorname{arity} F_i}$.