1°. $\operatorname{uncurry}(\operatorname{curry}(f)) = f$ for every $f \in Z^{\coprod_{i\in X} Y_i}$.
2°. $\operatorname{curry}(\operatorname{uncurry}(f)) = f$ for every $f \in \prod_{i\in X} Z^{Y_i}$.

3.8.2.2. *Functions with ordinal numbers of arguments.* Let Ord be the set of small ordinal numbers.

If X and Y are sets and n is an ordinal number, the set of functions taking n arguments on the set X and returning a value in Y is Y^{X^n}.

The set of all small functions taking ordinal numbers of arguments is $Y^{\bigcup_{n\in\mathrm{Ord}} X^n}$.

I will denote $\operatorname{OrdVar}(X) = \mho^{\bigcup_{n\in\mathrm{Ord}} X^n}$ and call it *ordinal variadic.* ("Var" in this notation is taken from the word *variadic* in the collocation *variadic function* used in computer science.)

3.8.3. On sums of ordinals. Let a be an ordinal-indexed family of ordinals.

PROPOSITION 235. $\coprod a$ with lexicographic order is a well-ordered set.

PROOF. Let S be non-empty subset of $\coprod a$.

Take $i_0 = \min \Pr_0 S$ and $x_0 = \min\left\{\frac{\Pr_1 y}{y\in S, y(0)=i_0}\right\}$ (these exist by properties of ordinals). Then $(i_0; x_0)$ is the least element of S. □

DEFINITION 236. $\sum a$ is the unique ordinal order-isomorphic to $\coprod a$. FiXme: For finite ordinals it is just a sum of natural numbers.

This ordinal exists and is unique because our set is well-ordered.

REMARK 237. An infinite sum of ordinals is not customary defined.

The *structured sum* $\bigoplus a$ of a is an order isomorphism from lexicographically ordered set $\coprod a$ into $\sum a$.

There exists (for a given a) exactly one structured sum, by properties of well-ordered sets.

OBVIOUS 238. $\sum a = \operatorname{im} \bigoplus a$.

THEOREM 239. $(\bigoplus a)(n; x) = \sum_{i\in n} a_i + x$.

PROOF. We need to prove that it is an order isomorphism. Let's prove it is an injection that is $m > n \Rightarrow \sum_{i\in m} a_i + x > \sum_{i\in n} a_i + x$ and $y > x \Rightarrow \sum_{i\in n} a_i + y > \sum_{i\in n} a_i + x$.

Really, if $m > n$ then $\sum_{i\in m} a_i + x \geq \sum_{i\in n+1} a_i + x > \sum_{i\in n} a_i + x$. The second formula is true by properties of ordinals.

Let's prove that it is a surjection. Let $r \in \sum a$. There exist $n \in \operatorname{dom} a$ and $x \in a_n$ such that $r = (\bigoplus a)(n; x)$. Thus $r = (\bigoplus a)(n; 0) + x = \sum_{i\in n} a_i + x$ because $(\bigoplus a)(n; 0) = \sum_{i\in n} a_i$ since $(n; 0)$ has $\sum_{i\in n} a_i$ predecessors. □

3.8.4. Ordinated product.

3.8.4.1. *Introduction.* *Ordinated product* defined below is a variation of Cartesian product, but is associative unlike Cartesian product. However, ordinated product unlike Cartesian product is defined not for arbitrary sets, but only for relations having ordinal numbers of arguments.

Let F indexed by an ordinal number be a small family of anchored relations.

3.8.4.2. *Concatenation.*

DEFINITION 240. Let z be an indexed by an ordinal number family of functions each taking an ordinal number of arguments. The *concatenation* of z is

$$\operatorname{concat} z = \operatorname{uncurry}(z) \circ \left(\bigoplus(\operatorname{dom} \circ z)\right)^{-1}.$$