Consequently, $(a \sqcup b) \setminus^* b \sqsubseteq \bigsqcap \left\{ \frac{z \in \mathfrak{A}}{a \sqsubseteq z} \right\} = a$. □

3.4. Several equal ways to express pseudodifference

THEOREM 202. For an atomistic co-brouwerian lattice $\mathfrak{A}$ and $a, b \in \mathfrak{A}$ the following expressions are always equal:

1°. $a \setminus^* b = \bigsqcap \left\{ \frac{z \in \mathfrak{A}}{a \sqsubseteq b \sqcup z} \right\}$ (quasidifference of a and b);
2°. $a \# b = \bigsqcup \left\{ \frac{z \in \mathfrak{A}}{z \sqsubseteq a \wedge z \sqcap b = \bot} \right\}$ (second quasidifference of a and b);
3°. $\bigsqcup (\operatorname{atoms} a \setminus \operatorname{atoms} b)$.

PROOF.

Proof of 1°=3°.

$$a \setminus^* b =$$
$$\left(\bigsqcup \operatorname{atoms} a \right) \setminus^* b = \quad \text{(theorem 133)}$$
$$\bigsqcup \left\{ \frac{A \setminus^* b}{A \in \operatorname{atoms} a} \right\} =$$
$$\bigsqcup \left\{ \frac{\left(\begin{cases} A & \text{if } A \notin \operatorname{atoms} b \\ \bot & \text{if } A \in \operatorname{atoms} b \end{cases} \right)}{A \in \operatorname{atoms} a} \right\} =$$
$$\bigsqcup \left\{ \frac{A}{A \in \operatorname{atoms} a, \notin \operatorname{atoms} b} \right\} =$$
$$\bigsqcup (\operatorname{atoms} a \setminus \operatorname{atoms} b).$$

Proof of 2°=3°. $a \setminus^* b$ is defined because our lattice is co-brouwerian. Taking the above into account, we have

$$a \setminus^* b =$$
$$\bigsqcup (\operatorname{atoms} a \setminus \operatorname{atoms} b) =$$
$$\bigsqcup \left\{ \frac{z \in \operatorname{atoms} a}{z \sqcap b = \bot.} \right\}$$

So $\bigsqcup \left\{ \frac{z \in \operatorname{atoms} a}{z \sqcap b = \bot.} \right\}$ is defined.

If $z \sqsubseteq a \wedge z \sqcap b = \bot$ then $z' = \bigsqcup \left\{ \frac{x \in \operatorname{atoms} z}{x \sqcap b = \bot} \right\}$ is defined. z' is a lower bound for $\left\{ \frac{z \in \operatorname{atoms} a}{x \sqcap b = \bot} \right\}$

Thus $z' \in \left\{ \frac{z \in \mathfrak{A}}{z \sqsubseteq a \wedge z \sqcap b = \bot} \right\}$ and so $\bigsqcup \left\{ \frac{z \in \operatorname{atoms} a}{z \sqcap b = \bot} \right\}$ is an upper bound of $\left\{ \frac{z \in \mathfrak{A}}{z \sqsubseteq a \wedge z \sqcap b = \bot} \right\}$.

If y is above every $z' \in \left\{ \frac{z \in \mathfrak{A}}{z \sqsubseteq a \wedge z \sqcap b = \bot} \right\}$ then y is above every $z \in \operatorname{atoms} a$ such that $z \sqcap b = \bot$ and thus y is above $\bigsqcup \left\{ \frac{z \in \operatorname{atoms} a}{z \sqcap b = \bot} \right\}$.

Thus $\bigsqcup \left\{ \frac{z \in \operatorname{atoms} a}{z \sqcap b = \bot} \right\}$ is least upper bound of

$$\left\{ \frac{z \in \mathfrak{A}}{z \sqsubseteq a \wedge z \sqcap b = \bot} \right\},$$