2°⇒4°. Let our semilattice be atomically separable. Let $a \sqsubset b$. Then $\operatorname{atoms} a \subset \operatorname{atoms} b$ and there exists $c \in \operatorname{atoms} b$ such that $c \notin \operatorname{atoms} a$. $c \neq \bot$ and $c \sqsubseteq b$, from which (taking into account that c is an atom) $c \sqsubseteq b$ and $c \sqcap a = \bot$. So our semilattice conforms to the formula (4).

4°⇒2°. Let formula (4) hold. Then for any elements $a \sqsubset b$ there exists $c \neq \bot$ such that $c \sqsubseteq b$ and $c \sqcap a = \bot$. Because $\mathfrak{A}$ is atomic there exists atom $d \sqsubseteq c$. $d \in \operatorname{atoms} b$ and $d \notin \operatorname{atoms} a$. So $\operatorname{atoms} a \neq \operatorname{atoms} b$ and $\operatorname{atoms} a \subseteq \operatorname{atoms} b$. Consequently $\operatorname{atoms} a \subset \operatorname{atoms} b$.

□

THEOREM 181. Any atomistic meet-semilattice with least element is separable.

PROOF. From the above. □

3.2. Free Stars

DEFINITION 182. An *upper set* is such a set $F \in \mathscr{P}\mathfrak{Z}$ that

$$\forall X \in F, Y \in \mathfrak{Z} : (Y \sqsupseteq X \Rightarrow Y \in F).$$

DEFINITION 183. Let $\mathfrak{A}$ be a poset. *Free stars* on $\mathfrak{A}$ are such $S \in \mathscr{P}\mathfrak{A}$ that the least element (if it exists) is not in S and for every $X, Y \in \mathfrak{A}$

$$\forall Z \in \mathfrak{A} : (Z \sqsupseteq X \wedge Z \sqsupseteq Y \Rightarrow Z \in S) \Leftrightarrow X \in S \vee Y \in S.$$

PROPOSITION 184. $S \in \mathscr{P}\mathfrak{A}$ where $\mathfrak{A}$ is a poset is a free star iff all of the following:

1°. The least element (if it exists) is not in S.
2°. $\forall Z \in \mathfrak{A} : (Z \sqsupseteq X \wedge Z \sqsupseteq Y \Rightarrow Z \in S) \Rightarrow X \in S \vee Y \in S$ for every $X, Y \in \mathfrak{A}$.
3°. S is an upper set.

PROOF.

⇒. 1° and 2° are obvious. Let prove that S is an upper set. Let $X \in S$ and $X \sqsubseteq Y \in \mathfrak{A}$. Then $X \in S \vee Y \in S$ and thus $\forall Z \in \mathfrak{A} : (Z \sqsupseteq X \wedge Z \sqsupseteq X \Rightarrow Z \in S)$ that is $\forall Z \in \mathfrak{A} : (Z \sqsupseteq X \Rightarrow Z \in S)$, and so $Y \in S$.

⇐. We need to prove that

$$\forall Z \in \mathfrak{A} : (Z \sqsupseteq X \wedge Z \sqsupseteq Y \Rightarrow Z \in S) \Leftarrow X \in S \vee Y \in S.$$

Let $X \in S \vee Y \in S$. Then $Z \sqsupseteq X \wedge Z \sqsupseteq Y \Rightarrow Z \in S$ for every $Z \in \mathfrak{A}$ because S is an upper set.

□

PROPOSITION 185. Let $\mathfrak{A}$ be a join-semilattice. $S \in \mathscr{P}\mathfrak{A}$ is a free star iff all of the following:

1°. The least element (if it exists) is not in S.
2°. $X \sqcup Y \in S \Rightarrow X \in S \vee Y \in S$ for every $X, Y \in \mathfrak{A}$.
3°. S is an upper set.

PROOF.

⇒. We need to prove only $X \sqcup Y \in S \Rightarrow X \in S \vee Y \in S$. Let $X \sqcup Y \in S$. Because S is an upper set, we have $\forall Z \in \mathfrak{A} : (Z \sqsupseteq X \sqcup Y \Rightarrow Z \in S)$ and thus $\forall Z \in \mathfrak{A} : (Z \sqsupseteq X \wedge Z \sqsupseteq Y \Rightarrow Z \in S)$ from which we conclude $X \in S \vee Y \in S$.

⇐. We need to prove $\forall Z \in \mathfrak{A} : (Z \sqsupseteq X \wedge Z \sqsupseteq Y \Rightarrow Z \in S) \Leftarrow X \in S \vee Y \in S$.
But it trivially follows from that S is an upper set.

□