8°⇒7°. Let $b \not\sqsubseteq a$. Then $a \sqcap b \sqsubset b$ that is $a' \sqsubset b$ where $a' = a \sqcap b$. Consequently $\exists c \in \mathfrak{A} \setminus \{\bot\} : (c \asymp a' \wedge c \sqsubseteq b)$. We have $c \sqcap a = c \sqcap b \sqcap a = c \sqcap a'$. So $c \sqsubseteq b$ and $c \sqcap a = \bot$. Thus Wallman's disjunction property holds.

□

PROPOSITION 174. Every boolean lattice is separable.

PROOF. Let $a, b \in \mathfrak{A}$ where $\mathfrak{A}$ is a boolean lattice an $a \neq b$. Then $a \sqcap \bar{b} \neq \bot$ or $\bar{a} \sqcap b \neq \bot$ because otherwise $a \sqcap \bar{b} = \bot$ and $a \sqcup \bar{b} = \top$ and thus $a = b$. Without loss of generality assume $a \sqcap \bar{b} \neq \bot$. Then $a \sqcap c \neq \bot$ and $b \sqcap c = \bot$ for $c = a \sqcap \bar{b} \neq \bot$. □

3.1.3. Atomically Separable Lattices.

PROPOSITION 175. "atoms" is a straight monotone map (for any meet-semilattice).

PROOF. Monotonicity is obvious. The rest follows from the formula

$$\operatorname{atoms}(a \sqcap b) = \operatorname{atoms} a \cap \operatorname{atoms} b$$

(corollary 97). □

DEFINITION 176. I will call *atomically separable* such a poset that "atoms" is an injection.

PROPOSITION 177. $\forall a, b \in \mathfrak{A} : (a \sqsubset b \Rightarrow \operatorname{atoms} a \subset \operatorname{atoms} b)$ iff $\mathfrak{A}$ is atomically separable for a poset $\mathfrak{A}$.

PROOF.

⇐. Obvious.

⇒. Let $a \neq b$ for example $a \not\sqsubseteq b$. Then $a \sqcap b \sqsubset a$; $\operatorname{atoms} a \supset \operatorname{atoms}(a \sqcap b) = \operatorname{atoms} a \cap \operatorname{atoms} b$ and thus $\operatorname{atoms} a \neq \operatorname{atoms} b$.

□

PROPOSITION 178. Any atomistic poset is atomically separable.

PROOF. We need to prove that $\operatorname{atoms} a = \operatorname{atoms} b \Rightarrow a = b$. But it is obvious because

$$a = \bigsqcup \operatorname{atoms} a \quad \text{and} \quad b = \bigsqcup \operatorname{atoms} b.$$

□

THEOREM 179. If a lattice with least element is atomic and separable then it is atomistic.

PROOF. Suppose the contrary that is $a \sqsupset \bigsqcup \operatorname{atoms} a$. Then, because our lattice is separable, there exists $c \in \mathfrak{A}$ such that $c \sqcap a \neq \bot$ and $c \sqcap \bigsqcup \operatorname{atoms} a = \bot$. There exists atom $d \sqsubseteq c$ such that $d \sqsubseteq c \sqcap a$. $d \sqcap \bigsqcup \operatorname{atoms} a \sqsubseteq c \sqcap \bigsqcup \operatorname{atoms} a = \bot$. But $d \in \operatorname{atoms} a$. Contradiction. □

THEOREM 180. Let $\mathfrak{A}$ be an atomic meet-semilattice with least element. Then the following statements are equivalent:

1°. $\mathfrak{A}$ is separable.
2°. $\mathfrak{A}$ is atomically separable.
3°. $\mathfrak{A}$ conforms to Wallman's disjunction property.
4°. $\forall a, b \in \mathfrak{A} : (a \sqsubset b \Rightarrow \exists c \in \mathfrak{A} \setminus \{\bot\} : (c \asymp a \wedge c \sqsubseteq b))$.

PROOF.

1°⇔3°⇔4°. Proved above.