PROPOSITION 69. If b is complementive to a then $(a \setminus b) \sqcup b = a$.

PROOF. Because $b \sqsubseteq a$ by the previous proposition. □

DEFINITION 70. Let $\mathfrak{A}$ be a bounded distributive lattice. The *complement* (denoted $\bar{a}$) of an element $a \in \mathfrak{A}$ is such $b \in \mathfrak{A}$ that $a \sqcap b = \bot$ and $a \sqcup b = \top$.

PROPOSITION 71. If $\mathfrak{A}$ is a bounded distributive lattice then $\bar{a} = \top \setminus a$.

PROOF. $b = \bar{a} \Leftrightarrow b \sqcap a = \bot \wedge b \sqcup a = \top \Leftrightarrow b \sqcap a = \bot \wedge \top \sqcup a = a \sqcup b \Leftrightarrow b = \top \setminus a$. □

COROLLARY 72. If $\mathfrak{A}$ is a bounded distributive lattice then exists no more than one complement of an element $a \in \mathfrak{A}$.

DEFINITION 73. An element of bounded distributive lattice is called *complemented* when its complement exists.

DEFINITION 74. A distributive lattice is a *complemented lattice* iff every its element is complemented.

PROPOSITION 75. For a distributive lattice $(a \setminus b) \setminus c = a \setminus (b \sqcup c)$ if $a \setminus b$ and $(a \setminus b) \setminus c$ are defined.

PROOF. $((a \setminus b) \setminus c) \sqcap c = \bot$; $((a \setminus b) \setminus c) \sqcup c = (a \setminus b) \sqcup c$; $(a \setminus b) \sqcap b = \bot$; $(a \setminus b) \sqcup b = a \sqcup b$.

We need to prove $((a \setminus b) \setminus c) \sqcap (b \sqcup c) = \bot$ and $((a \setminus b) \setminus c) \sqcup (b \sqcup c) = a \sqcup (b \sqcup c)$.

In fact,

$$\begin{aligned}((a \setminus b) \setminus c) \sqcap (b \sqcup c) &= \\ (((a \setminus b) \setminus c) \sqcap b) \sqcup (((a \setminus b) \setminus c) \sqcap c) &= \\ (((a \setminus b) \setminus c) \sqcap b) \sqcup \bot &= \\ ((a \setminus b) \setminus c) \sqcap b &\sqsubseteq \\ (a \setminus b) \sqcap b &= \bot,\end{aligned}$$

so $((a \setminus b) \setminus c) \sqcap (b \sqcup c) = \bot$;

$$\begin{aligned}&((a \setminus b) \setminus c) \sqcup (b \sqcup c) = \\ &(((a \setminus b) \setminus c) \sqcup c) \sqcup b = \\ &(a \setminus b) \sqcup c \sqcup b = \\ &((a \setminus b) \sqcup b) \sqcup c = \\ &a \sqcup b \sqcup c.\end{aligned}$$

□

2.1.8. Boolean lattices.

DEFINITION 76. A *boolean lattice* is a complemented distributive lattice.

The most important example of a boolean lattice is $\mathscr{P} A$ where A is a set, ordered by set inclusion.

THEOREM 77. (De Morgan's laws) For every elements a, b of a boolean lattice

1°. $\overline{a \sqcup b} = \bar{a} \sqcap \bar{b}$;
2°. $\overline{a \sqcap b} = \bar{a} \sqcup b$.

PROOF. We will prove only the first as the second is dual.

It is enough to prove that $a \sqcup b$ is a complement of $\bar{a} \sqcap \bar{b}$. Really:

$$(a \sqcup b) \sqcap (\bar{a} \sqcap \bar{b}) \sqsubseteq a \sqcap (\bar{a} \sqcap \bar{b}) = (a \sqcap \bar{a}) \sqcap \bar{b} = \bot \sqcap \bar{b} = \bot;$$

$$(a \sqcup b) \sqcup (\bar{a} \sqcap \bar{b}) = ((a \sqcup b) \sqcup \bar{a}) \sqcap ((a \sqcup b) \sqcup \bar{b}) \sqsupseteq (a \sqcup \bar{a}) \sqcap (b \sqcup \bar{b}) = \top \sqcap \top = \top.$$

Thus $(a \sqcup b) \sqcap (\bar{a} \sqcap \bar{b}) = \bot$ and $(a \sqcup b) \sqcup (\bar{a} \sqcap \bar{b}) = \top$. □