Really,

$$\begin{aligned} y \sqsupseteq \bigsqcup \bigcup S &\Leftrightarrow \\ \forall x \in \bigcup S : y \sqsupseteq x &\Leftrightarrow \\ \forall X \in S \forall x \in X : y \sqsupseteq x &\Leftrightarrow \\ \forall X \in S : y \sqsupseteq \bigsqcup X &\Leftrightarrow \\ y \sqsupseteq \bigsqcup \left\{ \frac{\bigsqcup X}{X \in S} \right\}&. \end{aligned}$$

□

2.1.6. Distributivity of lattices.

Definition 63. A *distributive* lattice is such lattice $\mathfrak{A}$ that for every $x, y, z \in \mathfrak{A}$

1°. $x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)$;
2°. $x \sqcup (y \sqcap z) = (x \sqcup y) \sqcap (x \sqcup z)$.

Theorem 64. For a lattice to be distributive it is enough just one of the conditions:

1°. $x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)$;
2°. $x \sqcup (y \sqcap z) = (x \sqcup y) \sqcap (x \sqcup z)$.

Proof.

$$\begin{aligned} (x \sqcup y) \sqcap (x \sqcup z) &= \\ ((x \sqcup y) \sqcap x) \sqcup ((x \sqcup y) \sqcap z) &= \\ x \sqcup ((x \sqcap z) \sqcup (y \sqcap z)) &= \\ (x \sqcup (x \sqcap z)) \sqcup (y \sqcap z) &= \\ x \sqcup (y \sqcap z)& \end{aligned}$$

(applied $x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)$ twice). □

2.1.7. Difference and complement.

Definition 65. Let $\mathfrak{A}$ be a distributive lattice with least element $\bot$. The *difference* (denoted $a \setminus b$) of elements a and b is such $c \in \mathfrak{A}$ that $b \sqcap c = \bot$ and $a \sqcup b = b \sqcup c$. I will call b *substractive* from a when $a \setminus b$ exists.

Theorem 66. If $\mathfrak{A}$ is a distributive lattice with least element $\bot$, there exists no more than one difference of elements a, b.

Proof. Let c and d be both differences $a \setminus b$. Then $b \sqcap c = b \sqcap d = \bot$ and $a \sqcup b = b \sqcup c = b \sqcup d$. So

$$c = c \sqcap (b \sqcup c) = c \sqcap (b \sqcup d) = (c \sqcap b) \sqcup (c \sqcap d) = \bot \sqcup (c \sqcap d) = c \sqcap d.$$

Similarly $d = d \sqcap c$. Consequently $c = c \sqcap d = d \sqcap c = d$. □

Definition 67. I will call b *complementive* to a iff there exists $c \in \mathfrak{A}$ such that $b \sqcap c = \bot$ and $b \sqcup c = a$.

Proposition 68. b is complementive to a iff b is substractive from a and $b \sqsubseteq a$.

Proof.

$\Leftarrow$. Obvious.
$\Rightarrow$. We deduce $b \sqsubseteq a$ from $b \sqcup c = a$. Thus $a \sqcup b = a = b \sqcup c$.

□