for every $x \in \mathfrak{A}$. Really, this follows from the chain of equivalences:

$$\begin{aligned} x \sqsupseteq (a \sqcup b) \sqcup c \Leftrightarrow \\ x \sqsupseteq a \sqcup b \wedge x \sqsupseteq c \Leftrightarrow \\ x \sqsupseteq a \wedge x \sqsupseteq b \wedge x \sqsupseteq c \Leftrightarrow \\ x \sqsupseteq a \wedge x \sqsupseteq b \sqcup c \Leftrightarrow \\ x \sqsupseteq a \sqcup (b \sqcup c). \end{aligned}$$

□

OBVIOUS 53. $a \not\asymp b$ iff $a \sqcap b$ is non-least, for every elements a, b of a meet-semilattice.

OBVIOUS 54. $a \equiv b$ iff $a \sqcup b$ is the greatest element, for every elements a, b of a join-semilattice.

2.1.5. Lattices and complete lattices.

DEFINITION 55. A *bounded* poset is a poset having both least and greatest elements.

DEFINITION 56. *Lattice* is a poset which is both join-semilattice and meet-semilattice.

DEFINITION 57. A *complete lattice* is a poset $\mathfrak{A}$ such that for every $X \in \mathscr{P}\mathfrak{A}$ both $\bigsqcup X$ and $\bigsqcap X$ exist.

OBVIOUS 58. Every complete lattice is a lattice.

PROPOSITION 59. Every complete lattice is a bounded poset.

PROOF. $\bigsqcup\emptyset$ is the least and $\bigsqcap\emptyset$ is the greatest element. □

THEOREM 60. Let $\mathfrak{A}$ be a poset.

1°. If $\bigsqcup X$ is defined for every $X \in \mathscr{P}\mathfrak{A}$, then $\mathfrak{A}$ is a complete lattice.
2°. If $\bigsqcap X$ is defined for every $X \in \mathscr{P}\mathfrak{A}$, then $\mathfrak{A}$ is a complete lattice.

PROOF. See [26] or any lattice theory reference. □

OBVIOUS 61. If $X \subseteq Y$ for some $X, Y \in \mathscr{P}\mathfrak{A}$ where $\mathfrak{A}$ is a complete lattice, then

1°. $\bigsqcup X \sqsubseteq \bigsqcup Y$;
2°. $\bigsqcap X \sqsupseteq \bigsqcap Y$.

PROPOSITION 62. If $S \in \mathscr{P}\mathscr{P}\mathfrak{A}$ then for every complete lattice $\mathfrak{A}$

1°. $\bigsqcup\bigcup S = \bigsqcup\left\{\frac{\bigsqcup X}{X \in S}\right\}$;
2°. $\bigsqcap\bigcup S = \bigsqcap\left\{\frac{\bigsqcap X}{X \in S}\right\}$.

PROOF. We will prove only the first as the second is dual.

By definition of joins, it is enough to prove $y \sqsupseteq \bigsqcup\bigcup S \Leftrightarrow y \sqsupseteq \bigsqcup\left\{\frac{\bigsqcup X}{X \in S}\right\}$.