DEFINITION 41. *Upper bounds* of a set $X$ is the set $\left\{ \frac{y \in \mathfrak{A}}{\forall x \in X : y \sqsupseteq x} \right\}$.

The dual notion:

DEFINITION 42. *Lower bounds* of a set $X$ is the set $\left\{ \frac{y \in \mathfrak{A}}{\forall x \in X : y \sqsubseteq x} \right\}$.

DEFINITION 43. *Join* $\bigsqcup X$ (also called *supremum* and denoted "$\sup X$") of a set $X$ is the least element of its upper bounds (if it exists).

DEFINITION 44. *Meet* $\bigsqcap X$ (also called *infimum* and denoted "$\inf X$") of a set $X$ is the greatest element of its lower bounds (if it exists).

We will write $b = \bigsqcup X$ when $b \in \mathfrak{A}$ is the join of $X$ or say that $\bigsqcup X$ does not exist if there are no such $b \in \mathfrak{A}$. (And dually for meets.)

EXERCISE 45. Provide an example of $\bigsqcup X \notin X$ for some set $X$ on some poset.

I will denote meets and joins for a specific poset $\mathfrak{A}$ as $\bigsqcap^{\mathfrak{A}}$ and $\bigsqcup^{\mathfrak{A}}$.

PROPOSITION 46.

1°. If $b$ is the greatest element of $X$ then $\bigsqcup X = b$.
2°. If $b$ is the least element of $X$ then $\bigsqcap X = b$.

PROOF. We will prove only the first as the second is dual.
Let $b$ be the greatest element of $X$. Then upper bounds of $X$ are $\left\{ \frac{y \in \mathfrak{A}}{y \sqsupseteq b} \right\}$. Obviously $b$ is the least element of this set, that is the join. □

DEFINITION 47. *Binary joins and meets* are defined by the formulas

$$x \sqcup y = \bigsqcup \{x, y\} \quad \text{and} \quad x \sqcup y = \bigsqcap \{x, y\}.$$

OBVIOUS 48. $\sqcup$ and $\sqcap$ are symmetric operations (whenever these are defined for given $x$ and $y$).

THEOREM 49.

1°. If $\bigsqcup X$ exists then $y \sqsupseteq \bigsqcup X \Leftrightarrow \forall x \in X : y \sqsupseteq x$.
2°. If $\bigsqcap X$ exists then $y \sqsubseteq \bigsqcap X \Leftrightarrow \forall x \in X : y \sqsubseteq x$.

PROOF. I will prove only the first as the second follows by duality. □

$y \sqsupseteq \bigsqcup X \Leftrightarrow y$ is an upper bound for $X \Leftrightarrow \forall x \in X : y \sqsupseteq x$.

COROLLARY 50.

1°. If $a \sqcup b$ exists then $y \sqsupseteq a \sqcup b \Leftrightarrow y \sqsupseteq a \wedge y \sqsupseteq b$.
2°. If $a \sqcap b$ exists then $y \sqsubseteq a \sqcap b \Leftrightarrow y \sqsubseteq a \wedge y \sqsubseteq b$.

### 2.1.4. Semilattices.

DEFINITION 51.

1°. A *join-semilattice* is a poset $\mathfrak{A}$ such that $a \sqcup b$ is defined for every $a, b \in \mathfrak{A}$.
2°. A *meet-semilattice* is a poset $\mathfrak{A}$ such that $a \sqcap b$ is defined for every $a, b \in \mathfrak{A}$.

THEOREM 52.

1°. The operation $\sqcup$ is associative for any join-semilattice.
2°. The operation $\sqcap$ is associative for any meet-semilattice.

PROOF. I will prove only the first as the second follows by duality.
We need to prove $(a \sqcup b) \sqcup c = a \sqcup (b \sqcup c)$ for every $a, b, c \in \mathfrak{A}$.
Taking into account the definition of join, it is enough to prove that

$$x \sqsupseteq (a \sqcup b) \sqcup c \Leftrightarrow x \sqsupseteq a \sqcup (b \sqcup c)$$