Proof. That $L\notin\operatorname{GR}\operatorname{ID}^{\mathsf{Strd}}_{\mathcal{A}[n]}$ if $L_k=0$ for some $k\in n$ is obvious. It remains to prove

$$L\cup\{(k;X\sqcup Y)\}\in\operatorname{GR}\operatorname{ID}^{\mathsf{Strd}}_{\mathcal{A}[n]}\Leftrightarrow L\cup\{(k;X)\}\in\operatorname{GR}\operatorname{ID}^{\mathsf{Strd}}_{\mathcal{A}[n]}\vee L\cup\{(k;Y)\}\in\operatorname{GR}\operatorname{ID}^{\mathsf{Strd}}_{\mathcal{A}[n]}.$$

It is equivalent to

$$\prod_{i\in n\setminus\{k\}} L_i\sqcap(X\sqcup Y)\not\asymp\mathcal{A}\Leftrightarrow\prod_{i\in n\setminus\{k\}} L_i\sqcap X\not\asymp\mathcal{A}\vee\prod_{i\in n\setminus\{k\}} L_i\sqcap Y\not\asymp\mathcal{A}.$$

Really, $\prod_{i\in n\setminus\{k\}} L_i\sqcap(X\sqcup Y)\not\asymp\mathcal{A}\Leftrightarrow\left(\prod_{i\in n\setminus\{k\}} L_i\sqcap X\right)\sqcup\left(\prod_{i\in n\setminus\{k\}} L_i\sqcap Y\right)\not\asymp\mathcal{A}\Leftrightarrow\prod_{i\in n\setminus\{k\}} L_i\sqcap X\not\asymp\mathcal{A}\vee\prod_{i\in n\setminus\{k\}} L_i\sqcap Y\not\asymp\mathcal{A}$. □

Proposition 44. Let $(\mathfrak{A};\mathfrak{Z})$ be a starrish filtrator over a complete meet infinite distributive lattice and $\mathcal{A}\in\mathfrak{A}$. Then $\operatorname{id}^{\mathsf{Strd}}_{\mathcal{A}[n]}$ is a staroid.

Proof. That $L\notin\operatorname{GR}\operatorname{id}^{\mathsf{Strd}}_{\mathcal{A}[n]}$ if $L_k=0$ for some $k\in n$ is obvious. It remains to prove

$$L\cup\{(k;X\sqcup Y)\}\in\operatorname{GR}\operatorname{id}^{\mathsf{Strd}}_{\mathcal{A}[n]}\Leftrightarrow L\cup\{(k;X)\}\in\operatorname{GR}\operatorname{id}^{\mathsf{Strd}}_{\mathcal{A}[n]}\vee L\cup\{(k;Y)\}\in\operatorname{GR}\operatorname{id}^{\mathsf{Strd}}_{\mathcal{A}[n]}.$$

It is equivalent to

$$\prod^{\mathfrak{Z}}_{i\in n\setminus\{k\}} L_i\sqcap(X\sqcup Y)\not\asymp\mathcal{A}\Leftrightarrow\prod^{\mathfrak{Z}}_{i\in n\setminus\{k\}} L_i\sqcap X\not\asymp\mathcal{A}\vee\prod^{\mathfrak{Z}}_{i\in n\setminus\{k\}} L_i\sqcap Y\not\asymp\mathcal{A}.$$

Really, $\prod^{\mathfrak{Z}}_{i\in n\setminus\{k\}} L_i\sqcap(X\sqcup Y)\not\asymp\mathcal{A}\Leftrightarrow\left(\prod^{\mathfrak{Z}}_{i\in n\setminus\{k\}} L_i\sqcap X\right)\sqcup\left(\prod^{\mathfrak{Z}}_{i\in n\setminus\{k\}} L_i\sqcap Y\right)\not\asymp\mathcal{A}\Leftrightarrow\prod^{\mathfrak{Z}}_{i\in n\setminus\{k\}} L_i\sqcap X\not\asymp\mathcal{A}\vee\prod^{\mathfrak{Z}}_{i\in n\setminus\{k\}} L_i\sqcap Y\not\asymp\mathcal{A}$. □

Proposition 45. Let $(\mathfrak{A};\mathfrak{Z})$ be a distributive lattice filtrator with least element and finitely join-closed core which is a join semilattice. $\operatorname{ID}^{\mathsf{Strd}}_{\mathcal{A}[n]}$ is a completary staroid for every $\mathcal{A}\in\mathfrak{A}$.

Proof. $\partial\mathcal{A}$ is a free star by theorem ??4.47.

$L_0\sqcup L_i\in\operatorname{GR}\operatorname{ID}^{\mathsf{Strd}}_{\mathcal{A}[n]}\Leftrightarrow\forall i\in n\colon(L_0\sqcup L_i)\,i\in\partial\mathcal{A}\Leftrightarrow\forall i\in n\colon L_0\,i\sqcup L_1\,i\in\partial\mathcal{A}\Leftrightarrow\forall i\in n\colon(L_0 i\in\partial\mathcal{A}\vee L_1 i\in\partial\mathcal{A})\Leftrightarrow\exists c\in\{0,1\}^n\forall i\in n\colon L_{c(i)}i\in\partial\mathcal{A}\Leftrightarrow\exists c\in\{0,1\}^n\colon(\lambda i\in n\colon L_{c(i)}i)\in\operatorname{GR}\operatorname{ID}^{\mathsf{Strd}}_{\mathcal{A}[n]}$. □

Lemma 46. $X\in\operatorname{GR}\operatorname{id}^{\mathsf{Strd}}_{\mathcal{A}[n]}\Leftrightarrow\operatorname{Cor}'\prod^{\mathfrak{A}}_{i\in n} X_i\not\asymp\mathcal{A}$ for a join-closed filtrator $(\mathfrak{A};\mathfrak{Z})$ such that both $\mathfrak{A}$ and $\mathfrak{Z}$ are complete lattices, provided that $\mathcal{A}\in\mathfrak{A}$.

Proof. $X\in\operatorname{GR}\operatorname{id}^{\mathsf{Strd}}_{\mathcal{A}[n]}\Leftrightarrow\prod^{\mathfrak{Z}}_{i\in n} X_i\not\asymp\mathcal{A}\Leftrightarrow\operatorname{Cor}'\prod^{\mathfrak{A}}_{i\in n} X_i\not\asymp\mathcal{A}$. □

Conjecture 47. $\operatorname{id}^{\mathsf{Strd}}_{\mathcal{A}[n]}$ is a completary staroid for every set-theoretic filter $\mathcal{A}$.

Proposition 48. Let each $(\mathfrak{A}_i;\mathfrak{Z}_i)$ for $i\in n$ (where n is an index set) is a finitely join-closed filtrator, such that each $\mathfrak{A}_i$ and each $\mathfrak{Z}_i$ are join-semilattices. If f is a completary staroid of the form $\mathfrak{A}$ then $\downdownarrows f$ is a completary staroid of the form $\mathfrak{Z}$. [TODO: Move this proposition (and note its corollary).]

Proof. $L_0\sqcup^{\mathfrak{Z}} L_1\in\operatorname{GR}\downdownarrows f\Leftrightarrow L_0\sqcup^{\mathfrak{Z}} L_1\in\operatorname{GR} f\Leftrightarrow L_0\sqcup^{\mathfrak{A}} L_1\in\operatorname{GR} f\Leftrightarrow\exists c\in\{0,1\}^n\colon(\lambda i\in n\colon L_{c(i)}i)\in\operatorname{GR} f\Leftrightarrow\exists c\in\{0,1\}^n\colon(\lambda i\in n\colon L_{c(i)}i)\in\operatorname{GR}\downdownarrows f$ for every $L_0,L_1\in\prod\mathfrak{Z}$. □

Conjecture 49. $\upuparrows\operatorname{id}^{\mathsf{Strd}}_{\mathcal{A}[n]}$ is a completary staroid if $\mathcal{A}$ is a filter on a set and n is an index set.

6.4 Special case of sets and filters

Proposition 50. $\uparrow^{\mathfrak{Z}^n}X\in\operatorname{GR}\operatorname{id}^{\mathsf{Strd}}_{a[n]}\Leftrightarrow\forall A\in a\colon\prod X\not\asymp\operatorname{id}_{A[n]}$ for every filter a on a powerset and index set n.

Proof. $\forall A\in a\colon\prod X\not\asymp\operatorname{id}_{A[n]}\Leftrightarrow\forall A\in a\colon\bigcap_{i\in n} X_i\cap A\neq\emptyset\Leftrightarrow\forall A\in a\colon\prod^{\mathfrak{P}}_{i\in n}\uparrow^{\mathfrak{Z}}X_i\not\asymp\uparrow A\Leftrightarrow\prod^{\mathfrak{P}}_{i\in n}(\uparrow^{\mathfrak{Z}^n}X_i)\not\asymp a\Leftrightarrow\prod^{\mathfrak{P}}_{i\in n}(\uparrow^{\mathfrak{Z}^n}X)_i\not\asymp a\Leftrightarrow\uparrow^{\mathfrak{Z}^n}X\in\operatorname{GR}\operatorname{id}_{a[n]}$. □