Proof.

1. $\bigsqcup^{\mathfrak{Z}} T \in \Downarrow S \Leftrightarrow \bigsqcup^{\mathfrak{Z}} T \in S \Leftrightarrow \bigsqcup^{\mathfrak{A}} T \in S \Leftrightarrow T \cap S \neq \emptyset \Leftrightarrow T \cap \Downarrow S \neq \emptyset$ for every $T \in \mathscr{P}\mathfrak{Z}$; $0 \notin \Downarrow S$ is obvious.

2. There exists a principal filter $\mathcal{F}$ such that $S = \partial \mathcal{F}$.

   $\bigsqcup^{\mathfrak{A}} T \in \Uparrow S \Leftrightarrow \operatorname{up} \bigsqcup^{\mathfrak{A}} T \in S \Leftrightarrow \forall K \in \operatorname{up} \bigsqcup^{\mathfrak{A}} T \colon K \in \partial \mathcal{F} \Leftrightarrow \forall K \in \operatorname{up} \bigsqcup^{\mathfrak{A}} T \colon K \not\asymp \mathcal{F} \Leftrightarrow \bigsqcup^{\mathfrak{A}} T \not\asymp \mathcal{F} \Leftrightarrow \bigsqcup^{\mathfrak{A}} T \in \star \mathcal{F} \Leftrightarrow \exists \mathcal{K} \in T \colon \mathcal{K} \in \star \mathcal{F} \Leftrightarrow \exists \mathcal{K} \in T \colon \mathcal{K} \not\asymp \mathcal{F} \Leftrightarrow \exists \mathcal{K} \in T \forall K \in \operatorname{up} \mathcal{K} \colon K \not\asymp \mathcal{F} \Leftrightarrow \exists \mathcal{K} \in T \forall K \in \operatorname{up} \mathcal{K} \colon K \in \partial \mathcal{F} \Leftrightarrow \exists \mathcal{K} \in T \colon \operatorname{up} \mathcal{K} \subseteq S \Leftrightarrow \exists \mathcal{K} \in T \colon \mathcal{K} \in \Uparrow S \Leftrightarrow T \cap \Uparrow S \neq \emptyset$.

   $0 \in \Uparrow S \Leftrightarrow \operatorname{up} 0 \subseteq S \Leftrightarrow 0 \in S$ what is false. □

Corollary 28. If S is a complete free star on $\mathfrak{F}$ then $\Downarrow S$ is a complete free star on $\mathfrak{P}$, provided that $\mathfrak{Z}$ is a complete lattice.

5 Complete staroids and multifuncoids

Definition 29. Consider an indexed family $(\mathfrak{A}_i; \mathfrak{Z}_i)$ of filtrators. A pre-staroid f of the form $\prod \mathfrak{Z}$ is *complete* in argument $k \in \operatorname{arity} f$ when $(\operatorname{val} f)_k L$ is a complete free star for every $L \in \prod_{i \in (\operatorname{arity} f) \setminus \{k\}} \mathfrak{Z}_i$.

Definition 30. Consider an indexed family $(\mathfrak{A}_i; \mathfrak{Z}_i)$ of filtrators and pre-multifuncoid f is of the form $\prod \mathfrak{A}$. Then f is *complete* in argument $k \in \operatorname{arity} f$ iff $\langle f \rangle_k L \in \mathfrak{Z}_k$ for every family $L \in \prod_{i \in (\operatorname{arity} f) \setminus \{k\}} \mathfrak{Z}_i$.

Proposition 31. Consider an indexed family $(\mathfrak{F}_i; \mathfrak{Z}_i)$ of primary filtrators over boolean lattices. Let f be a pre-multifuncoid of the form $\mathfrak{A}$ and $k \in \operatorname{arity} f$. The following are equivalent:

1. Pre-multifuncoid f is complete in argument k.
2. Pre-staroid $\Downarrow [f]$ is complete in argument k.

Proof. $L \in \operatorname{GR}[f] \Leftrightarrow L_i \not\asymp \langle f \rangle_i L|_{(\operatorname{dom} L) \setminus \{i\}}$;

$(\operatorname{val} \Downarrow [f])_k L = \partial \langle f \rangle_k L$ by the theorem ??17.81.

So $(\operatorname{val} \Downarrow [f])_k L$ is a complete free star iff $\langle f \rangle_k L \in \mathfrak{Z}_k$ (proposition 22) for every $L \in \prod_{i \in (\operatorname{arity} f) \setminus \{k\}} \mathfrak{Z}_i$. □

Example 32. Consider funcoid $f = \operatorname{id}^{\mathsf{FCD}(U)}$. It is obviously complete in each its two arguments. Then $[f]$ is not complete in each of its two arguments because $(\mathcal{X}; \mathcal{Y}) \in [f] \Leftrightarrow \mathcal{X} \not\asymp \mathcal{Y}$ what does not generate a complete free star if one of the arguments (say $\mathcal{X}$) is a fixed nonprincipal filter.

Theorem 33. Consider a semifiltered, star-separable, down-aligned filtrator $(\mathfrak{A}; \mathfrak{Z})$ with finitely meet closed and separable core where $\mathfrak{Z}$ is a complete boolean lattice and both $\mathfrak{Z}$ and $\mathfrak{A}$ are atomistic lattices.

Let f be a multifuncoid of the aforementioned form. Let $k, l \in \operatorname{arity} f$ and $k \neq l$. The following are equivalent:

1. f is complete in the argument k.
2. $\langle f \rangle_l (L \cup \{(k; \bigsqcup X)\}) = \bigsqcup_{x \in X} \langle f \rangle_l (L \cup \{(k; x)\})$ for every $X \in \mathscr{P}\mathfrak{Z}_k$, $L \in \prod_{i \in (\operatorname{arity} f) \setminus \{k, l\}} \mathfrak{Z}_i$.
3. $\langle f \rangle_l (L \cup \{(k; \bigsqcup X)\}) = \bigsqcup_{x \in X} \langle f \rangle_l (L \cup \{(k; x)\})$ for every $X \in \mathscr{P}\mathfrak{A}_k$, $L \in \prod_{i \in (\operatorname{arity} f) \setminus \{k, l\}} \mathfrak{Z}_i$.

Proof.

(3)⇒(2). Obvious.

(2)⇒(1). Let $Y \in \mathfrak{Z}$.

$\bigsqcup X \not\asymp \langle f \rangle_k (L \cup \{(l; Y)\}) \Leftrightarrow Y \not\asymp \langle f \rangle_l (L \cup \{(k; \bigsqcup X)\}) \Leftrightarrow Y \not\asymp \bigsqcup_{x \in X} \langle f \rangle_l (L \cup \{(k; x)\}) \Leftrightarrow$ (proposition 4.144??) $\Leftrightarrow \exists x \in X \colon Y \not\asymp \langle f \rangle_l (L \cup \{(k; x)\}) \Leftrightarrow \exists x \in X \colon x \not\asymp \langle f \rangle_k (L \cup (l; Y))$.