2. $\partial\mathcal{F}$ is a complete free star on $\mathfrak{Z}$.
3. $\star\mathcal{F}$ is a complete free star on $\mathfrak{F}$.

Proof.

(1)⇒(2). That $\partial\mathcal{F}$ does not contain the least element is obvious. That $\partial\mathcal{F}$ is an upper set is obvious. So it remains to apply theorem 4.53??.

(2)⇒(3). That $\star\mathcal{F}$ does not contain the least element is obvious. That $\star\mathcal{F}$ is an upper set is obvious. So it remains to apply theorem 4.53??.

(3)⇒(1). Apply theorem 4.53??. □

Corollary 23. For a filter $\mathcal{F}\in\mathfrak{F}$ on a complete atomic boolean lattice the following conditions are equivalent:

1. $\mathcal{F}\in\mathfrak{P}$.
2. $\partial\mathcal{F}$ is a complete free star on $\mathfrak{P}$.
3. $\star\mathcal{F}$ is a complete free star on $\mathfrak{F}$.

Theorem 24. Let $\mathfrak{Z}$ be a boolean lattice. For any set $S\in\mathscr{P}\mathfrak{P}$ there exists a principal filter $\mathcal{A}$ such that $\partial\mathcal{A}=S$ iff S is a complete free star (on $\mathfrak{P}$).

Proof.

⇒. From the previous theorem.

⇐. $0^{\mathfrak{P}}\notin S$ and $\bigsqcup T\in S\Leftrightarrow T\cap S\neq\emptyset\Leftrightarrow\exists X\in T: X\in S$. Take $\mathcal{A}=\{\overline{X}\mid X\in\mathfrak{P}\setminus S\}$. We will prove that $\mathcal{A}$ is a principal filter. That $\mathcal{A}$ is a filter follows from properties of free stars. It remains to show that $\mathcal{A}$ is a principal filter. It follows from the following equivalence:

$\bigsqcap^{\mathfrak{P}}\mathcal{A}\in\mathcal{A}\Leftrightarrow\overline{\bigsqcup^{\mathfrak{P}}\langle\neg\rangle\mathcal{A}}\in\mathcal{A}\Leftrightarrow\bigsqcup^{\mathfrak{P}}\langle\neg\rangle\mathcal{A}\notin S\Leftrightarrow\neg\exists X\in\langle\neg\rangle\mathcal{A}: X\in S\Leftrightarrow\forall X\in\langle\neg\rangle\mathcal{A}: X\notin S\Leftrightarrow\forall X\in\mathcal{A}: X\in\mathcal{A}\Leftrightarrow 1$. □

Proposition 25.

1. If S is a free star on $\mathfrak{A}$ then $\downarrow\downarrow S$ is a free star on $\mathfrak{Z}$, provided that $\mathfrak{Z}$ is a join-semilattice and the filtrator $(\mathfrak{A};\mathfrak{Z})$ is down-aligned and with finitely join-closed core.
2. If S is a free star on $\mathfrak{P}$ then $\uparrow\uparrow S$ is a free star on $\mathfrak{F}$, provided that $\mathfrak{Z}$ is a boolean lattice.

Proof.

1. $X\sqcup^{\mathfrak{Z}}Y\in\downarrow\downarrow S\Leftrightarrow X\sqcup^{\mathfrak{Z}}Y\in S\Leftrightarrow X\sqcup^{\mathfrak{A}}Y\in S\Leftrightarrow X\in S\vee Y\in S\Leftrightarrow X\in\downarrow\downarrow S\vee Y\in\downarrow\downarrow S$ for every $X,Y\in\mathfrak{Z}$; $0\notin\downarrow\downarrow S$ is obvious.
2. There exists a filter $\mathcal{F}$ such that $S=\partial\mathcal{F}$. For every filters $\mathcal{X},\mathcal{Y}\in\mathfrak{F}$
   $\mathcal{X}\sqcup^{\mathfrak{A}}\mathcal{Y}\in\uparrow\uparrow S\Leftrightarrow\operatorname{up}(\mathcal{X}\sqcup^{\mathfrak{A}}\mathcal{Y})\subseteq S\Leftrightarrow\forall K\in\operatorname{up}(\mathcal{X}\sqcup^{\mathfrak{F}}\mathcal{Y}): K\in\partial\mathcal{F}\Leftrightarrow\forall K\in\operatorname{up}(\mathcal{X}\sqcup^{\mathfrak{F}}\mathcal{Y}): K\not\asymp\mathcal{F}\Leftrightarrow\mathcal{X}\sqcup^{\mathfrak{F}}\mathcal{Y}\not\asymp\mathcal{F}\Leftrightarrow\mathcal{X}\sqcup^{\mathfrak{F}}\mathcal{Y}\in\star\mathcal{F}\Leftrightarrow\mathcal{X}\in\star\mathcal{F}\vee\mathcal{Y}\in\star\mathcal{F}\Leftrightarrow\mathcal{X}\not\asymp\mathcal{F}\vee\mathcal{Y}\not\asymp\mathcal{F}\Leftrightarrow\forall X\in\operatorname{up}\mathcal{X}: X\not\asymp\mathcal{F}\vee\forall Y\in\operatorname{up}\mathcal{Y}: Y\not\asymp\mathcal{F}\Leftrightarrow\forall X\in\operatorname{up}\mathcal{X}: X\in\partial\mathcal{F}\vee\forall Y\in\operatorname{up}\mathcal{Y}: Y\in\partial\mathcal{F}\Leftrightarrow\operatorname{up}\mathcal{X}\subseteq S\vee\operatorname{up}\mathcal{Y}\subseteq S\Leftrightarrow\mathcal{X}\in\uparrow\uparrow S\vee\mathcal{Y}\in\uparrow\uparrow S$;
   $0\in\uparrow\uparrow S\Leftrightarrow\operatorname{up}0\subseteq S\Leftrightarrow 0\in S$ what is false. □

Corollary 26. If S is a free star on $\mathfrak{F}$ then $\downarrow\downarrow S$ is a free star on $\mathfrak{P}$, provided that $\mathfrak{P}$ is a join-semilattice.

Proposition 27.

1. If S is a complete free star on $\mathfrak{A}$ then $\downarrow\downarrow S$ is a complete free star on $\mathfrak{Z}$, provided that $\mathfrak{Z}$ is a complete lattice and the filtrator $(\mathfrak{A};\mathfrak{Z})$ is down-aligned and with join-closed core.
2. If S is a complete free star on $\mathfrak{P}$ then $\uparrow\uparrow S$ is a complete free star on $\mathfrak{F}$, provided that $\mathfrak{Z}$ is a boolean lattice.