2. $\bigsqcup T \in S \Rightarrow T \cap S \neq \emptyset$ for every $T \in \mathscr{P}S$.
3. S is an upper set.

Proof.

⇒. We need to prove only $\bigsqcup T \in S \Rightarrow T \cap S \neq \emptyset$. Let $\bigsqcup T \in S$. Because S is an upper set, we have $\forall X \in T: Z \sqsupseteq X \Rightarrow Z \sqsupseteq \bigsqcup T \Rightarrow Z \in S$ from which we conclude $T \cap S \neq \emptyset$.

⇐. We need to prove only $\forall Z \in \mathfrak{A}: (\forall X \in T: Z \sqsupseteq X \Rightarrow Z \in S) \Rightarrow T \cap S \neq \emptyset$.
Really, if $\forall Z \in \mathfrak{A}: (\forall X \in T: Z \sqsupseteq X \Rightarrow Z \in S)$ then $\bigsqcup T \in S$ and thus $\bigsqcup T \in S \Rightarrow T \cap S \neq \emptyset$. □

Proposition 13. Let $\mathfrak{A}$ be a complete lattice. $S \in \mathscr{P}\mathfrak{A}$ is a complete free star iff the least element (if it exists) is not in S and for every $T \in \mathscr{P}\mathfrak{A}$

$$\bigsqcup T \in S \Leftrightarrow T \cap S \neq \emptyset.$$

Proof.

⇒. We need to prove only $\bigsqcup T \in S \Leftarrow T \cap S \neq \emptyset$ what follows from that S is an upper set.

⇐. We need to prove only that S is an upper set. To prove this we can use the fact that S is a free star. □

4.1.1 Completely starrish posets

Definition 14. I will call a poset *completely starrish* when the full star $\star a$ is a free star for every element a of this poset.

Obvious 15. Every completely starrish poset is starrish.

Proposition 16. Every complete join infinite distributive lattice is starrish.

Proof. Let $\mathfrak{A}$ be a join infinite distributive lattice, $a \in \mathfrak{A}$. Obviously $0 \notin \star a$ (if 0 exists); obviously $\star a$ is an upper set. If $\bigsqcup T \in \star a$, then $(\bigsqcup T) \sqcap a$ is non-least that is $\bigsqcup \langle a \sqcap \rangle T$ is non-least what is equivalent to $a \sqcap x$ being non-least for some $x \in T$ that is $x \in \star a$. □

Theorem 17. If $\mathfrak{A}$ is a completely starrish complete lattice lattice then

$$\operatorname{atoms} \bigsqcup T = \bigcup \langle \operatorname{atoms} \rangle T.$$

for every $T \in \mathscr{P}\mathfrak{A}$.

Proof. For every atom c we have: $c \in \operatorname{atoms} \bigsqcup T \Leftrightarrow c \not\asymp \bigsqcup T \Leftrightarrow \bigsqcup T \in \star c \Leftrightarrow \exists X \in T: X \in \star c \Leftrightarrow \exists X \in T: X \not\asymp c \Leftrightarrow \exists X \in T: c \in \operatorname{atoms} X \Leftrightarrow c \in \bigcup \langle \operatorname{atoms} \rangle T$. □

4.2 More on free stars and complete free stars

Obvious 18. $\partial \mathcal{F} = \downdownarrows \star \mathcal{F}$ for an element $\mathcal{F}$ of down-aligned finitely meet closed filtrator.

Corollary 19. $\partial \mathcal{F} = \downdownarrows \star \mathcal{F}$ for every filter $\mathcal{F}$ on a poset.

Proposition 20. $\star \mathcal{F} = \upuparrows \partial \mathcal{F}$ for an element $\mathcal{F}$ of a filtrator with separable core.

Proof. $\mathcal{X} \in \upuparrows \partial \mathcal{F} \Leftrightarrow \operatorname{up} \mathcal{X} \subseteq \partial \mathcal{F} \Leftrightarrow \forall X \in \mathcal{X}: X \not\asymp \mathcal{F} \Leftrightarrow \mathcal{X} \not\asymp \mathcal{F} \Leftrightarrow \mathcal{X} \in \star \mathcal{F}$. □

Corollary 21. $\star \mathcal{F} = \upuparrows \partial \mathcal{F}$ for every filter $\mathcal{F}$ on a distributive lattice with least element.

Proposition 22. For a semifiltered, star-separable, down-aligned filtrator $(\mathfrak{A}; \mathfrak{Z})$ with finitely meet closed and separable core where $\mathfrak{Z}$ is a complete boolean lattice and both $\mathfrak{Z}$ and $\mathfrak{A}$ are atomistic lattices the following conditions are equivalent for any $\mathcal{F} \in \mathfrak{A}$:

1. $\mathcal{F} \in \mathfrak{Z}$.