$\rho_0 \circ x = \rho_0 \circ y \Rightarrow \lambda i \in n$: $(\mathrm{Src}\, x = \mathrm{Src}\, y \wedge \mathrm{Dst}\, x = \mathrm{Dst}\, y) \Rightarrow \rho_1 \prod^{(C)} x = \mathsf{FCD}(\mathrm{StarHom}(\lambda i \in \mathrm{dom}\, x\colon \mathrm{Src}\, x_i); \mathrm{StarHom}(\lambda i \in \mathrm{dom}\, x\colon \mathrm{Dst}\, x_i)) = \mathsf{FCD}(\mathrm{StarHom}(\lambda i \in \mathrm{dom}\, y\colon \mathrm{Src}\, y_i); \mathrm{StarHom}(\lambda i \in \mathrm{dom}\, y\colon \mathrm{Dst}\, y_i)) = \rho_1 \prod^{(C)} y$.

We have proved that it is a pre-quasi-cartesian function.

Next prove that it is a quasi-cartesian function, that is

$$\left(\prod^{(C)}\right)|_{\{x \in X_0^{\mathrm{dom}\,\mathfrak{A}} \mid \rho_0 \circ x = \mathfrak{A}\} \setminus \mathrm{ZC}_0}$$

is an injection for every indexed family $\mathfrak{A}$ of forms. Let $x \in \{x \in X_0^{\mathrm{dom}\,\mathfrak{A}} \mid \rho_0 \circ x = \mathfrak{A}\} \setminus \mathrm{ZC}_0$. To prove that it is an injection we will restore the value of x from $\prod^{(C)} x$.

$\left\langle \prod^{(C)} x \right\rangle \{p\} = \mathrm{StarComp}(\{p\}; x)$ for every $p \in \mho^n$.

$L \in \mathrm{GR}\, \mathrm{StarComp}(\{p\}; x) \Leftrightarrow \forall i \in n\colon p_i\, x_i\, L_i \Leftrightarrow \forall i \in n\colon L_i \in \langle x_i \rangle \{p_i\}$ for every $L \in \mho^n$.

Thus $\mathrm{GR}\, \mathrm{StarComp}(\{p\}; x) = \prod_{i \in n} \langle x_i \rangle \{p_i\}$.

Since $x_i \neq 0$ there exist p such that $\langle x_i \rangle \{p_i\} \neq 0$. Take $k \in n$, $p'_i = p_i$ for $i \neq k$ and $p_k = q$ for an arbitrary value q; then

$$\langle x_k \rangle \{q\} = \Pr_k \prod_{i \in n} \langle x_i \rangle \{p'_i\} = \Pr_k \mathrm{GR}\, \mathrm{StarComp}(\{p'\}; x) = \Pr_k \mathrm{GR} \left\langle \prod^{(C)} x \right\rangle \{p'\}.$$

So the value of x can be restored from $\prod^{(C)} x$ by this formula.

It remained to prove that it is with injective aggregation.

We have $\Upsilon F = (\mathrm{StarHom}(\lambda i \in \mathrm{dom}\, f\colon F_{i,0}); \mathrm{StarHom}(\lambda i \in \mathrm{dom}\, f\colon F_{i,1}))$ for every form F.

It is really an injection because $\mathrm{StarHom}(-)$ are disjoint. □

Theorem 37. *Cross-composition product (for small indexed families of pointfree funcoids between separable atomic posets with least elements and atomistic posets) is a quasi-cartesian function (with injective aggregation) from the quasi-cartesian situation $\mathfrak{S}_0$ of pointfree funcoids over posets with least elements to the quasi-cartesian situation $\mathfrak{S}_1$ of pointfree funcoids over posets with least elements.*

Proof. First prove that it is a pre-quasi-cartesian function. We need to show that for every small indexed families x, y of pointfree funcoids:

1. $x \in \mathrm{ZC}_0 \Leftrightarrow \prod^{(C)} x = Z_1\left(\rho_1 \prod^{(C)} x\right)$;

2. $\rho_0 \circ x = \rho_0 \circ y \Rightarrow \rho_1 \prod^{(C)} x = \rho_1 \prod^{(C)} y$;

$\prod^{(C)} x = Z_1\left(\rho_1 \prod^{(C)} x\right) \Leftrightarrow \prod^{(C)} x = Z_1(\mathsf{FCD}(\mathrm{StarHom}(\lambda i \in \mathrm{dom}\, x\colon \mathrm{Src}\, x_i); \mathrm{StarHom}(\lambda i \in \mathrm{dom}\, x\colon \mathrm{Dst}\, x_i))) \Leftrightarrow \prod^{(C)} x = 0^{\mathsf{FCD}(\mathrm{StarHom}(\lambda i \in \mathrm{dom}\, x: \mathrm{Src}\, x_i); \mathrm{StarHom}(\lambda i \in \mathrm{dom}\, x: \mathrm{Dst}\, x_i))} \Leftrightarrow \forall a \in \mathrm{StarHom}(\lambda i \in \mathrm{dom}\, x\colon \mathrm{Src}\, x_i)\colon \left\langle \prod^{(C)} x \right\rangle a = 0^{\mathrm{StarHom}(\lambda i \in \mathrm{dom}\, x: \mathrm{Dst}\, x_i)} \Leftrightarrow \forall a \in \mathrm{StarHom}(\lambda i \in \mathrm{dom}\, x\colon \mathrm{Src}\, x_i)\colon \mathrm{StarComp}(a; x) = 0^{\mathrm{StarHom}(\lambda i \in \mathrm{dom}\, x: \mathrm{Dst}\, x_i)} \Leftrightarrow \forall a \in \mathrm{StarHom}(\lambda i \in \mathrm{dom}\, x\colon \mathrm{Src}\, x_i)\colon \mathrm{GR}\, \mathrm{StarComp}(a; x) = \emptyset$;

$\forall a \in \mathrm{StarHom}(\lambda i \in \mathrm{dom}\, x\colon \mathrm{Src}\, x_i)\colon \mathrm{GR}\, \mathrm{StarComp}(a; x) = \emptyset \Leftarrow x \in \mathrm{ZC}_0$.