Proof. $\operatorname{card}\langle f^{-1}\rangle\{y\}\geqslant 1$ is obvious. It remains to show that $y\,[f^{-1}]\,a\wedge y\,[f^{-1}]\,b\Rightarrow a=b$ for every a and b. Really, let $y\,[f^{-1}]\,a\wedge y\,[f^{-1}]\,b$. Then $y=fa$ and $y=fb$ and thus $a=b$ because

$y\in\langle f\rangle(\{x\in X_0^{\operatorname{dom}\mathfrak{A}}\mid\rho_0\circ x=\mathfrak{A}\}\setminus\mathrm{ZC}_0)$ and $f|_{\{x\in X_0^{\operatorname{dom}\mathfrak{A}}\mid\rho_0\circ x=\mathfrak{A}\}\setminus\mathrm{ZC}_0}$ is an injection. □

Fix quasi-cartesian situations $\mathfrak{S}_A$, $\mathfrak{S}_B$, $\mathfrak{S}_C$ and quasi-cartesian functions f: $\mathfrak{S}_A\rightarrow\mathfrak{S}_B$ and g: $\mathfrak{S}_A\rightarrow\mathfrak{S}_C$ such that $\operatorname{dom}f=\operatorname{dom}g$. Let $\mathfrak{A}$ is a small indexed family of forms. [TODO: Check below formulations (it is possible that I've done little errors confusing A, B, and C).]

For a small indexed family $\mathfrak{A}$ of forms let:

$\varphi_{\mathfrak{A}}=g\circ\mathrm{id}_{\{x\in X_A^{\operatorname{dom}\mathfrak{A}}\mid\rho_A\circ x=\mathfrak{A}\}}\circ f^{-1}$.

Obvious 15. $\varphi_{\mathfrak{A}}=g|_{\{x\in X_A^{\operatorname{dom}\mathfrak{A}}\mid\rho_A\circ x=\mathfrak{A}\}}\circ f^{-1}=g\circ\left(f|_{\{x\in X_A^{\operatorname{dom}\mathfrak{A}}\mid\rho_A\circ x=\mathfrak{A}\}}\right)^{-1}$.

Proposition 16. *$\varphi_{\mathfrak{A}}$ is a function and $\operatorname{dom}\varphi_{\mathfrak{A}}=\langle f\rangle\{x\in X_A^{\operatorname{dom}\mathfrak{A}}\mid\rho_A\circ x=\mathfrak{A}\}$ and for every $y\in\operatorname{dom}\varphi_{\mathfrak{A}}$ we have*

$$\varphi_{\mathfrak{A}}\,y=\begin{cases}Z_C(\Upsilon_g\mathfrak{A}) & \text{if } y=Z_B(\Upsilon_f\mathfrak{A});\\ g\,f^{-1}\,y & \text{if } y\neq Z_B(\Upsilon_f\mathfrak{A}).\end{cases}$$

Proof. It follows from the previous proposition. □

Theorem 17. $\varphi_{\mathfrak{A}}=g\circ f^{-1}|_{\langle f\rangle\{x\in X_A^{\operatorname{dom}\mathfrak{A}}\mid\rho_A\circ x=\mathfrak{A}\}}$.

Proof. If $y\in(\langle f\rangle\{x\in X_A^{\operatorname{dom}\mathfrak{A}}\mid\rho_A\circ x=\mathfrak{A}\})\setminus\{Z_B(\Upsilon_f\mathfrak{A})\}$ then $\operatorname{card}\langle f^{-1}\rangle\{y\}=1$ and thus $\langle f\rangle\{y\}\in\{x\in X_B^{\operatorname{dom}\mathfrak{A}}\mid\rho_B\circ x=\mathfrak{A}\}\setminus\mathrm{ZC}_B$. Consequently

$$\langle\varphi_{\mathfrak{A}}\rangle\{y\}=\langle g\circ f^{-1}\rangle\{y\}=\left\langle g\circ f^{-1}|_{\langle f\rangle\{x\in X_A^{\operatorname{dom}\mathfrak{A}}\mid\rho_A\circ x=\mathfrak{A}\}}\right\rangle\{y\}.$$

$\langle\varphi_{\mathfrak{A}}\rangle\{Z_B(\Upsilon_f\mathfrak{A})\}=Z_C(\Upsilon_g\mathfrak{A})$ and

$$\begin{aligned}\left\langle g\circ f^{-1}|_{\langle f\rangle\{x\in X_A^{\operatorname{dom}\mathfrak{A}}\mid\rho_A\circ x=\mathfrak{A}\}}\right\rangle\{Z_C(\Upsilon_g\mathfrak{A})\} &=\\ \langle g\circ f^{-1}\rangle\{Z_C(\Upsilon_g\mathfrak{A})\} &=\\ \langle g\rangle(\{x\in X_A^{\operatorname{dom}\mathfrak{A}}\mid\rho_A\circ x=\mathfrak{A}\}\cap\mathrm{ZC}_A) &=\\ Z_C(\Upsilon_g\mathfrak{A}).\end{aligned}$$

Thus $\langle\varphi_{\mathfrak{A}}\rangle\{y\}=\left\langle g\circ f^{-1}|_{\langle f\rangle\{x\in X_A^{\operatorname{dom}\mathfrak{A}}\mid\rho_A\circ x=\mathfrak{A}\}}\right\rangle\{y\}$ for every $y\in\langle f\rangle\{x\in X_A^{\operatorname{dom}\mathfrak{A}}\mid\rho_A\circ x=\mathfrak{A}\}$. □

Theorem 18. *$\varphi_{\mathfrak{A}}$ is a bijection*

$$\langle f\rangle\{x\in X_A^{\operatorname{dom}\mathfrak{A}}\mid\rho_A\circ x=\mathfrak{A}\}\rightarrow\langle g\rangle\{x\in X_A^{\operatorname{dom}\mathfrak{A}}\mid\rho_A\circ x=\mathfrak{A}\}.$$

Proof. That $\varphi_{\mathfrak{A}}$ is a surjection

$$\langle f\rangle\{x\in X_A^{\operatorname{dom}\mathfrak{A}}\mid\rho_0\circ x=\mathfrak{A}\}\rightarrow\langle g\rangle\{x\in X_A^{\operatorname{dom}\mathfrak{A}}\mid\rho_0\circ x=\mathfrak{A}\}$$

follows from a proposition above and symmetry. To prove that it is an injection is enough to show that:

1. $g\,f^{-1}\,y\neq Z_C(\Upsilon_g\mathfrak{A})$ if $y\neq Z_B(\Upsilon_f\mathfrak{A})$ for every $y\in\operatorname{dom}\varphi_{\mathfrak{A}}$.