Theorem 29. $\prod^{(L)}$ together with $\bigotimes$ is a (partial) product in the category cont($\mathcal{C}$).

Proof. Obvious.

Check http://math.stackexchange.com/questions/102632/how-to-check-whether-it-is-a-direct-product/102677#102677 □

4 On duality

We will consider duality where both the category $\mathcal{C}$ and orders on Mor-sets are replaced with their dual. I will denote $A \overset{\text{dual}}{\longleftrightarrow} B$ when two formulas A and B are dual with this duality.

Proposition 30. $f \in \mathrm{C}(\mu; \nu) \overset{\text{dual}}{\longleftrightarrow} f^{\dagger} \in \mathrm{C}(\nu^{\dagger}; \mu^{\dagger})$.

Proof. $f \in \mathrm{C}(\mu; \nu) \Leftrightarrow f \circ \mu \sqsubseteq \nu \circ f \overset{\text{dual}}{\longleftrightarrow} \mu^{\dagger} \circ f^{\dagger} \sqsupseteq f^{\dagger} \circ^{\dagger} \nu^{-1} \Leftrightarrow f^{\dagger} \in \mathrm{C}(\nu^{\dagger}; \mu^{\dagger})$. □

f is entirely defined$\Leftrightarrow f^{\dagger} \circ f \sqsupseteq 1_{\mathrm{Src}\, f} \overset{\text{dual}}{\longleftrightarrow} f^{\dagger} \circ f \sqsubseteq 1_{\mathrm{Src}\, f} \Leftrightarrow f$ is injective$\Leftrightarrow f^{\dagger}$ is monovalued.

f is monovalued$\Leftrightarrow f \circ f^{\dagger} \sqsubseteq 1_{\mathrm{Dst}\, f} \overset{\text{dual}}{\longleftrightarrow} f \circ f^{\dagger} \sqsupseteq 1_{\mathrm{Dst}\, f} \Leftrightarrow f$ is surjective$\Leftrightarrow f^{\dagger}$ is entirely defined.

5 General coproduct in partially ordered dagger category

The below is the dual of the above, proofs are omitted as they are dual.

Let ι_i [TODO: What is i?] are entirely defined monovalued morphisms to an object Z.

Let $\iota_i \overset{\text{dual}}{\longleftrightarrow} \pi_i$ that is $\iota_i = (\pi_i)^{\dagger}$. We have the above equivalent to π_i being monovalued and entirely defined.

5.1 Supremum coproduct

Let $\mathcal{C}$ be a dagger category, each Mor-set of which is a complete lattice (having order agreed with the dagger).

We will designate some morphisms as *principal* and require that principal morphisms are both metacomplete and co-metacomplete. (For a particular example of the category Rel, all morphisms are considered principal.)

Let $\coprod^{(Q)} X$ be an object for each indexed family X of objects.

Let ι be a partial function mapping elements $X \in \operatorname{dom} \iota$ (which consists of small indexed families of objects of $\mathcal{C}$) to indexed families $X_i \rightarrow \coprod^{(Q)} X$ of principal morphisms (called *injections*) for every $i \in \operatorname{dom} X$.

Definition 31. *Supremum coproduct* $\coprod^{(L)} F$ (such that ι is defined at $\lambda j \in n: \mathrm{Dst}\, F_j$ and $\lambda j \in n: \mathrm{Src}\, F_j$) is defined by the formula

$$\coprod^{(L)} F = \bigsqcup_{i \in \operatorname{dom} F} \left(\iota_i^{\lambda j \in n: \mathrm{Src}\, F_j} \circ F_i^{\dagger} \circ \left(\iota_i^{\lambda j \in n: \mathrm{Dst}\, F_j} \right)^{\dagger} \right).$$

This formula can be (over)simplified to:

$$\coprod^{(L)} F = \bigsqcup_{i \in \operatorname{dom} F} \left(\iota_i^{\mathrm{Src} \circ F} \circ F_i^{\dagger} \circ \left(\iota_i^{\mathrm{Dst} \circ F} \right)^{\dagger} \right).$$

Remark 32. $\iota_i^{\lambda j \in n: \mathrm{Src}\, F_j} \circ F_i \circ \left(\iota_i^{\lambda j \in n: \mathrm{Dst}\, F_j} \right)^{\dagger} \in \mathrm{Mor}\left(\coprod_{j \in n}^{(Q)} \mathrm{Src}\, F_j; \coprod_{j \in n}^{(Q)} \mathrm{Dst}\, F_j \right)$ are properly defined and have the same sources and destination (whenever $i \in \operatorname{dom} F$ is), thus the meet in the formulas is properly defined.