Remark 14. $\left(\pi_i^{\lambda j\in n:\operatorname{Dst}F_j}\right)^\dagger\circ F_i\circ\pi_i^{\lambda j\in n:\operatorname{Src}F_j}\in\operatorname{Mor}\left(\prod^{(Q)}_{j\in n}\operatorname{Src}F_j;\prod^{(Q)}_{j\in n}\operatorname{Dst}F_j\right)$ are properly defined and have the same sources and destination (whenever $i\in\operatorname{dom}F$ is), thus the meet in the formulas is properly defined.

Remark 15. Thus

$$F_0\times^{(L)}F_1=\left(\left(\pi_0^{(\operatorname{Dst}F_0;\operatorname{Dst}F_1)}\right)^\dagger\circ F_0\circ\pi_0^{(\operatorname{Src}F_0;\operatorname{Src}F_1)}\right)\sqcap\left(\left(\pi_1^{(\operatorname{Dst}F_0;\operatorname{Dst}F_1)}\right)^\dagger\circ F_1\circ\pi_1^{(\operatorname{Src}F_0;\operatorname{Src}F_1)}\right)$$

that is product is defined by a pure algebraic formula.

Proposition 16. $\prod^{(L)}F=\max\left\{\Phi\in\operatorname{Mor}\left(\prod^{(Q)}_{j\in n}\operatorname{Src}F_j;\prod^{(Q)}_{j\in n}\operatorname{Dst}F_j\right)\mid\forall i\in n:\Phi\sqsubseteq\left(\pi_i^{\lambda j\in n:\operatorname{Dst}F_j}\right)^\dagger\circ F_i\circ\pi_i^{\lambda j\in n:\operatorname{Src}F_j}\right\}$.

Proof. By definition of meet on a complete lattice. □

Corollary 17. $\prod^{(L)}F=\bigsqcup\left\{\Phi\in\operatorname{Mor}\left(\prod^{(Q)}_{j\in n}\operatorname{Src}F_j;\prod^{(Q)}_{j\in n}\operatorname{Dst}F_j\right)\mid\forall i\in n:\Phi\sqsubseteq\left(\pi_i^{\lambda j\in n:\operatorname{Dst}F_j}\right)^\dagger\circ F_i\circ\pi_i^{\lambda j\in n:\operatorname{Src}F_j}\right\}$.

Theorem 18. Let π_i^X be metamonovalued morphisms. If $S\in\mathscr{P}(\operatorname{Mor}(A_0;B_0)\times\operatorname{Mor}(A_1;B_1))$ for some sets A_0, B_0, A_1, B_1 then

$$\bigsqcap\left\{a\times^{(L)}b\mid(a;b)\in S\right\}=\bigsqcap\operatorname{dom}S\times^{(L)}\bigsqcap\operatorname{im}S.$$

Proof. $\bigsqcap\{a\times b\mid(a;b)\in S\}=\bigsqcap\left\{\left(\left(\pi_0^{(\operatorname{Dst}a;\operatorname{Dst}b)}\right)^\dagger\circ a\circ\pi_0^{(\operatorname{Src}a;\operatorname{Src}b)}\right)\sqcap\left(\left(\pi_1^{(\operatorname{Dst}a;\operatorname{Dst}b)}\right)^\dagger\circ b\circ\pi_1^{(\operatorname{Src}a;\operatorname{Src}b)}\right)\mid(a;b)\in S\right\}=\bigsqcap\left\{\left(\pi_0^{(\operatorname{Dst}a;\operatorname{Dst}b)}\right)^\dagger\circ a\circ\pi_0^{(\operatorname{Src}a;\operatorname{Src}b)}\mid a\in\operatorname{dom}S\right\}\sqcap\bigsqcap\left\{\left(\pi_1^{(\operatorname{Dst}a;\operatorname{Dst}b)}\right)^\dagger\circ b\circ\pi_1^{(\operatorname{Src}a;\operatorname{Src}b)}\mid b\in\operatorname{im}S\right\}=\left(\left(\pi_0^{(\operatorname{Dst}a;\operatorname{Dst}b)}\right)^\dagger\circ\bigsqcap\{a\mid a\in\operatorname{dom}S\}\circ\pi_0^{(\operatorname{Src}a;\operatorname{Src}b)}\right)\sqcap\left(\left(\pi_1^{(\operatorname{Dst}a;\operatorname{Dst}b)}\right)^\dagger\circ\bigsqcap\{b\mid b\in\operatorname{im}S\}\circ\pi_1^{(\operatorname{Src}a;\operatorname{Src}b)}\right)=\left(\left(\pi_0^{(\operatorname{Dst}a;\operatorname{Dst}b)}\right)^\dagger\circ(\bigsqcap\operatorname{dom}S)\circ\pi_0^{(\operatorname{Src}a;\operatorname{Src}b)}\right)\sqcap\left(\left(\pi_1^{(\operatorname{Dst}a;\operatorname{Dst}b)}\right)^\dagger\circ(\bigsqcap\operatorname{im}S)\circ\pi_1^{(\operatorname{Src}a;\operatorname{Src}b)}\right)=\bigsqcap\operatorname{dom}S\times\bigsqcap\operatorname{im}S.$ □

Corollary 19. $\left(a_0\times^{(L)}b_0\right)\sqcap\left(a_1\times^{(L)}b_1\right)=(a_0\sqcap a_1)\times^{(L)}(b_0\sqcap b_1)$.

Corollary 20. $a_0\times^{(L)}b_0\not\asymp a_1\times^{(L)}b_1\Leftrightarrow a_0\not\asymp a_1\wedge b_0\not\asymp b_1$.

3.2 Infimum product for endomorphisms

Let F is an indexed family of endomorphisms of $\mathcal{C}$.

I will denote $\operatorname{Ob}f$ the object (source and destination) of an endomorphism f.

Let also π_i^X be a monovalued entirely defined morphism (for each $i\in\operatorname{dom}F$).

Then $\prod^{(L)}F=\bigsqcap_{i\in\operatorname{dom}F}\left(\left(\pi_i^{\lambda j\in n:\operatorname{Ob}F_j}\right)^\dagger\circ F_i\circ\pi_i^{\lambda j\in n:\operatorname{Ob}F_j}\right)$ (if π is defined at $\lambda j\in n:\operatorname{Ob}F_j$).

Abbreviate $\pi_i=\pi_i^{\lambda j\in n:\operatorname{Ob}F_j}$.

So $\prod^{(L)}F=\bigsqcap_{i\in\operatorname{dom}F}((\pi_i)^\dagger\circ F_i\circ\pi_i)$.

$\prod^{(L)}F=\max\left\{\Phi\in\operatorname{End}\left(\prod^{(Q)}_{j\in n}\operatorname{Ob}F_j\right)\mid\forall i\in n:\Phi\sqsubseteq(\pi_i)^\dagger\circ F_i\circ\pi_i\right\}$.

Taking into account that π_i is a monovalued entirely defined morphism, we get:

Obvious 21. $\prod^{(L)}F=\max\left\{\Phi\in\operatorname{End}\left(\prod^{(Q)}_{j\in n}\operatorname{Ob}F_j\right)\mid\forall i\in n:\pi_i\in\mathrm{C}(\Phi;F_i)\right\}$.

Remark 22. The above formula may allow to define the product for non-dagger categories (but only for endomorphisms). In this writing I don't introduce a notation for this, however.

Corollary 23. $\pi_i\in\mathrm{C}\left(\prod^{(L)}F;F_i\right)$ for every $i\in\operatorname{dom}F$.