2. A morphism $(f;\mathfrak{A};\mathfrak{B};\mathcal{A};\mathcal{B})$ of the category of pointfree funcoid triples is injective iff the funcoid f is injective.

Theorem 96. Let $\mathfrak{A}$ is an atomistic meet-semilattice, $\mathfrak{B}$ is a bounded meet-semilattice. The following statements are equivalent for every $f\in\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$:

1. f is monovalued.
2. $\forall a\in\operatorname{atoms}^{\mathfrak{A}}\colon \langle f\rangle a\in\operatorname{atoms}^{\mathfrak{B}}\cup\{0^{\mathfrak{B}}\}$.
3. $\forall i,j\in\mathfrak{A}\colon \langle f^{-1}\rangle(i\cap^{\mathfrak{B}} j)=\langle f^{-1}\rangle i\cap^{\mathfrak{A}}\langle f^{-1}\rangle j$.

Proof.

(2)⇒(3). Let $a\in\operatorname{atoms}^{\mathfrak{A}}$, $\langle f\rangle a=b$. Then because $b\in\operatorname{atoms}^{\mathfrak{B}}\cup\{0^{\mathfrak{B}}\}$

$$(i\cap^{\mathfrak{B}} j)\cap^{\mathfrak{B}} b\neq 0^{\mathfrak{B}}\Leftrightarrow i\cap^{\mathfrak{B}} b\neq 0^{\mathfrak{B}}\wedge j\cap^{\mathfrak{B}} b\neq 0^{\mathfrak{B}};$$
$$a\,[f]\,(i\cap^{\mathfrak{B}} j)\Leftrightarrow a\,[f]\,i\wedge a\,[f]\,j;$$
$$(i\cap^{\mathfrak{B}} j)\,[f^{-1}]\,a\Leftrightarrow i\,[f^{-1}]\,a\wedge j\,[f^{-1}]\,a;$$
$$a\cap^{\mathfrak{A}}\langle f^{-1}\rangle(i\cap^{\mathfrak{B}} j)\neq 0^{\mathfrak{A}}\Leftrightarrow a\cap^{\mathfrak{A}}\langle f^{-1}\rangle i\neq 0^{\mathfrak{A}}\wedge a\cap^{\mathfrak{A}}\langle f^{-1}\rangle j\neq 0^{\mathfrak{A}};$$
$$a\cap^{\mathfrak{A}}\langle f^{-1}\rangle(i\cap^{\mathfrak{B}} j)\neq 0^{\mathfrak{A}}\Leftrightarrow a\cap^{\mathfrak{A}}\langle f^{-1}\rangle i\cap^{\mathfrak{A}}\langle f^{-1}\rangle j\neq 0^{\mathfrak{A}};$$
$$\langle f^{-1}\rangle(i\cap^{\mathfrak{B}} j)=\langle f^{-1}\rangle i\cap^{\mathfrak{B}}\langle f^{-1}\rangle j.$$

(3)⇒(1). $\langle f^{-1}\rangle a\cap^{\mathfrak{A}}\langle f^{-1}\rangle b=\langle f^{-1}\rangle(a\cap^{\mathfrak{B}} b)=\langle f^{-1}\rangle 0^{\mathfrak{B}}=0^{\mathfrak{A}}$ for every two distinct a, $b\in\operatorname{atoms}^{\mathfrak{B}}$. This is equivalent to $\neg(\langle f^{-1}\rangle a\,[f]\,b)$; $b\cap^{\mathfrak{B}}\langle f\rangle\langle f^{-1}\rangle a=0$; $b\cap^{\mathfrak{B}}\langle f\circ f^{-1}\rangle a=0^{\mathfrak{B}}$; $\neg(a\,[f\circ f^{-1}]\,b)$. So $a\,[f\circ f^{-1}]\,b\Rightarrow a=b$ for every a, $b\in\operatorname{atoms}^{\mathfrak{B}}$. This is possible only (corollary 53) when $f\circ f^{-1}\subseteq 1^{\mathfrak{B}}$.

¬(2)⇒¬(1). Suppose $\langle f\rangle a\notin\operatorname{atoms}^{\mathfrak{B}}\cup\{0^{\mathfrak{B}}\}$ for some $a\in\operatorname{atoms}^{\mathfrak{A}}$. Then there exist two atoms $p\neq q$ such that $\langle f\rangle a\supseteq p\wedge\langle f\rangle a\supseteq q$. Consequently $p\cap^{\mathfrak{B}}\langle f\rangle a\neq 0^{\mathfrak{B}}$; $a\cap^{\mathfrak{A}}\langle f^{-1}\rangle p\neq 0^{\mathfrak{A}}$; $a\subseteq\langle f^{-1}\rangle p$; $\langle f\circ f^{-1}\rangle p=\langle f\rangle\langle f^{-1}\rangle p\supseteq\langle f\rangle a\supseteq q$; $\langle f\circ f^{-1}\rangle p\nsubseteq p$ and $\langle f\circ f^{-1}\rangle p\neq 0^{\mathfrak{B}}$. So it cannot be $f\circ f^{-1}\subseteq 1^{\mathfrak{B}}$. □

Theorem 97. Let $(\mathfrak{B};\mathfrak{Z}_1)$ is a primary filtrator over a meet-semilattice with greatest element and $(\mathfrak{A};\mathfrak{Z}_0)$ is a primary filtrator over a boolean lattice. A pointfree funcoid $f\in\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$ is monovalued iff

$$\forall I,J\in\mathfrak{Z}_0\colon \langle f^{-1}\rangle(I\cap^{\mathfrak{Z}_1} J)=\langle f^{-1}\rangle I\cap^{\mathfrak{A}}\langle f^{-1}\rangle J.$$

Proof. $\mathfrak{A}$ and $\mathfrak{B}$ are complete lattices (corollary 8 in [3]).

$(\mathfrak{B};\mathfrak{Z}_1)$ is a filtrator with separable core by the theorem 37 in [3].

$(\mathfrak{B};\mathfrak{Z}_1)$ is finitely meet-closed by the theorem 29 in [3].

$\mathfrak{A}$ is an atomistic lattice by the theorem 48 in [3].

We are under conditions of the previous theorem.

⇒. Obvious.

⇐. $\langle f^{-1}\rangle(\mathcal{I}\cap^{\mathfrak{B}}\mathcal{J})=\bigcap^{\mathfrak{A}}\langle\langle f^{-1}\rangle\rangle\operatorname{up}^{(\mathfrak{B};\mathfrak{Z}_1)}(\mathcal{I}\cap^{\mathfrak{B}}\mathcal{J})=\bigcap^{\mathfrak{A}}\langle\langle f^{-1}\rangle\rangle\{I\cap^{\mathfrak{Z}_1} J\mid I\in\operatorname{up}\mathcal{I}, J\in\operatorname{up}\mathcal{J}\}=\bigcap^{\mathfrak{A}}\{\langle f^{-1}\rangle(I\cap^{\mathfrak{Z}_1} J)\mid I\in\operatorname{up}\mathcal{I}, J\in\operatorname{up}\mathcal{J}\}=\bigcap^{\mathfrak{A}}\{\langle f^{-1}\rangle I\cap^{\mathfrak{A}}\langle f^{-1}\rangle J\mid I\in\operatorname{up}\mathcal{I}, J\in\operatorname{up}\mathcal{J}\}=\bigcap^{\mathfrak{A}}\{\langle f^{-1}\rangle I\mid I\in\operatorname{up}\mathcal{I}\}\cap^{\mathfrak{A}}\bigcap^{\mathfrak{A}}\{\langle f^{-1}\rangle J\mid J\in\operatorname{up}\mathcal{J}\}=\langle f^{-1}\rangle\mathcal{I}\cap^{\mathfrak{A}}\langle f^{-1}\rangle\mathcal{J}$ (used theorem 25, theorem 34 in [3], theorem 15). □

3.14 Elements closed regarding a pointfree funcoid

Let $\mathfrak{A}$ is a poset with least element. Let $f\in\mathsf{FCD}(\mathfrak{A};\mathfrak{A})$.

Definition 98. Let's call *closed* regarding a pointfree funcoid f such element $a\in\mathfrak{A}$ that $\langle f\rangle a\subseteq a$.

Proposition 99. If i and j are closed (regarding a pointfree funcoid $f\in\mathsf{FCD}(\mathfrak{A};\mathfrak{A})$), S is a set of closed elements (regarding f), then

1. $i\cup^{\mathfrak{A}} j$ is a closed element, if $\mathfrak{A}$ is a separable starrish join-semilattice;