Theorem 84. Let $(\mathfrak{A}; \mathfrak{Z}_0)$ is semifiltered, star-separable, down-aligned filtrator with finitely meet closed, join-closed, and separable core, where $\mathfrak{Z}_0$ is a complete boolean lattice and both $\mathfrak{Z}_0$ and $\mathfrak{A}$ are atomistic lattices.

Let $(\mathfrak{B}; \mathfrak{Z}_1)$ is a star-separable filtrator.

The following conditions are equivalent for every pointfree funcoid $f \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$:

1. f^{-1} is co-complete;
2. $\forall S \in \mathscr{P}\mathfrak{A}, J \in \mathfrak{Z}_1: (\bigcup^{\mathfrak{A}} S\, [f]\, J \Rightarrow \exists \mathcal{I} \in S: \mathcal{I}\, [f]\, J)$;
3. $\forall S \in \mathscr{P}\mathfrak{Z}_0, J \in \mathfrak{Z}_1: (\bigcup^{\mathfrak{Z}_0} S\, [f]\, J \Rightarrow \exists I \in S: I\, [f]\, J)$;
4. f is complete;
5. $\forall S \in \mathscr{P}\mathfrak{Z}_0: \langle f\rangle \bigcup^{\mathfrak{Z}_0} S = \bigcup^{\mathfrak{B}} \langle\langle f\rangle\rangle S$.

Proof. First note that the theorem 53 in [3] applies to the filtrator $(\mathfrak{A}; \mathfrak{Z}_0)$.

(3)⇒(1). For every $S \in \mathscr{P}\mathfrak{Z}_0$, $J \in \mathfrak{Z}_1$

$$\bigcup^{\mathfrak{Z}_0} S \cap^{\mathfrak{A}} \langle f^{-1}\rangle J \neq 0^{\mathfrak{A}} \Rightarrow \exists I \in S: I \cap^{\mathfrak{A}} \langle f^{-1}\rangle J \neq 0^{\mathfrak{A}}, \tag{9}$$

consequently by the theorem 53 in [3] we have $\langle f^{-1}\rangle J \in \mathfrak{Z}_0$.

(1)⇒(2). For every $S \in \mathscr{P}\mathfrak{A}$, $J \in \mathfrak{Z}_1$ we have $\langle f^{-1}\rangle J \in \mathfrak{Z}_0$, consequently the formula (9) is true. From this follows (2).

(2)⇒(4). Let $\langle f\rangle \bigcup^{\mathfrak{Z}_0} S$ and $\bigcup^{\mathfrak{B}} \langle\langle f\rangle\rangle S$ are defined. $J \cap^{\mathfrak{B}} \langle f\rangle \bigcup^{\mathfrak{A}} S \neq 0^{\mathfrak{B}} \Leftrightarrow \bigcup^{\mathfrak{A}} S\, [f]\, J \Leftrightarrow \exists \mathcal{I} \in S: \mathcal{I}\, [f]\, J \Leftrightarrow \exists \mathcal{I} \in S: J \cap^{\mathfrak{B}} \langle f\rangle \mathcal{I} \neq 0^{\mathfrak{B}} \Leftrightarrow J \cap^{\mathfrak{B}} \bigcup^{\mathfrak{B}} \langle\langle f\rangle\rangle S \neq 0^{\mathfrak{B}}$ (used the theorem 53 in [3]). Thus $\langle f\rangle \bigcup^{\mathfrak{A}} S = \bigcup^{\mathfrak{B}} \langle\langle f\rangle\rangle S$ by star-separability of $(\mathfrak{B}; \mathfrak{Z}_1)$.

(5)⇒(3). Let $\langle f\rangle \bigcup^{\mathfrak{Z}_0} S$ is defined. Then $\bigcup^{\mathfrak{B}} \langle\langle f\rangle\rangle S$ is also defined because $\langle f\rangle \bigcup^{\mathfrak{Z}_0} S = \bigcup^{\mathfrak{B}} \langle\langle f\rangle\rangle S$. Then $\bigcup^{\mathfrak{Z}_0} S\ [f]\ J \Leftrightarrow J \cap^{\mathfrak{B}} \langle f\rangle \bigcup^{\mathfrak{Z}_0} S \neq 0^{\mathfrak{B}} \Leftrightarrow J \cap^{\mathfrak{B}} \bigcup^{\mathfrak{B}} \langle\langle f\rangle\rangle S \neq 0^{\mathfrak{B}}$ what by the theorem 53 in [3] equivalent to $\exists I \in S: J \cap^{\mathfrak{B}} \langle f\rangle I \neq 0^{\mathfrak{B}}$ that is $\exists I \in S: I\, [f]\, J$.

(2)⇒(3), (4)⇒(5). By join-closedness of the core of $(\mathfrak{A}; \mathfrak{Z}_0)$. □

Theorem 85. Let $(\mathfrak{A}; \mathfrak{Z}_0)$ and $(\mathfrak{B}; \mathfrak{Z}_1)$ are primary filtrators over boolean lattices and $\mathfrak{Z}_0$ is a complete boolean lattice. If R is a set of co-complete pointfree funcoids in $\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ then $\bigcup^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} R$ is a co-complete pointfree funcoid.

Proof. First, conditions of the theorem 84 apply.

Let R is a set of co-complete pointfree funcoids. Then for every $X \in \mathfrak{Z}_0$

$$\left\langle \bigcup^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} R \right\rangle X = \bigcup^{\mathfrak{Z}_1} \{ \langle f\rangle X \mid f \in R \} \in \mathfrak{Z}_1$$

(used the theorem 30). □

Let $\mathfrak{A}$ and $\mathfrak{B}$ are posets with least elements. I will denote $\mathrm{Compl}\mathsf{FCD}\,(\mathfrak{A}; \mathfrak{B})$ and $\mathrm{CoCompl}\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ the sets of complete and co-complete funcoids correspondingly from a poset $\mathfrak{A}$ to a poset $\mathfrak{B}$ with least elements.

Proposition 86.

1. Let $f \in \mathrm{Compl}\mathsf{FCD}\,(\mathfrak{A}; \mathfrak{B})$ and $g \in \mathrm{Compl}\mathsf{FCD}\,(\mathfrak{B}; \mathfrak{C})$ where $\mathfrak{A}$ and $\mathfrak{C}$ are posets with least elements and $\mathfrak{B}$ is a complete lattice. Then $g \circ f \in \mathrm{Compl}\mathsf{FCD}\,(\mathfrak{A}; \mathfrak{C})$.
2. Let $f \in \mathrm{CoCompl}\mathsf{FCD}\,(\mathfrak{A}; \mathfrak{B})$ and $g \in \mathrm{CoCompl}\mathsf{FCD}\,(\mathfrak{B}; \mathfrak{C})$ where $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ are posets with least elements and $(\mathfrak{A}; \mathfrak{Z}_0)$, $(\mathfrak{B}; \mathfrak{Z}_1)$, $(\mathfrak{C}; \mathfrak{Z}_2)$ are filtrators. Then $g \circ f \in \mathrm{CoCompl}\mathsf{FCD}\,(\mathfrak{A}; \mathfrak{C})$.

Proof.

1. Let $\bigcup^{\mathfrak{A}} S$ and $\bigcup^{\mathfrak{C}} \langle\langle g \circ f\rangle\rangle S$ are defined. Then

$$\langle g \circ f\rangle \bigcup^{\mathfrak{A}} S = \langle g\rangle \langle f\rangle \bigcup^{\mathfrak{A}} S = \langle g\rangle \bigcup^{\mathfrak{B}} \langle\langle f\rangle\rangle S = \bigcup^{\mathfrak{C}} \langle\langle g\rangle\rangle \langle\langle f\rangle\rangle S = \bigcup^{\mathfrak{C}} \langle\langle g \circ f\rangle\rangle S.$$