$h \overset{\mathrm{def}}{=} I_{\mathcal{B}}^{\mathsf{FCD}(\mathfrak{B})} \circ f \circ I_{\mathcal{A}}^{\mathsf{FCD}(\mathfrak{A})}$. For every $\mathcal{X} \in \mathfrak{A}$

$$\langle h \rangle \mathcal{X} = \left\langle I_{\mathcal{B}}^{\mathsf{FCD}(\mathfrak{B})} \right\rangle \langle f \rangle \left\langle I_{\mathcal{A}}^{\mathsf{FCD}(\mathfrak{A})} \right\rangle \mathcal{X} = \mathcal{B} \cap^{\mathfrak{B}} \langle f \rangle (\mathcal{A} \cap^{\mathfrak{A}} \mathcal{X}).$$

From this, as easy to show, $h \subseteq f$ and $h \subseteq \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}$. If $g \subseteq f \wedge g \subseteq \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}$ for a $g \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ then $\operatorname{dom} g \subseteq \mathcal{A}$, $\operatorname{im} g \subseteq \mathcal{B}$,

$$\langle g \rangle \mathcal{X} = \mathcal{B} \cap^{\mathfrak{B}} \langle g \rangle (\mathcal{A} \cap^{\mathfrak{A}} \mathcal{X}) \subseteq \mathcal{B} \cap^{\mathfrak{B}} \langle f \rangle (\mathcal{A} \cap^{\mathfrak{A}} \mathcal{X}) = \left\langle I_{\mathcal{B}}^{\mathsf{FCD}(\mathfrak{B})} \right\rangle \langle f \rangle \left\langle I_{\mathcal{A}}^{\mathsf{FCD}(\mathfrak{A})} \right\rangle \mathcal{X} = \langle h \rangle \mathcal{X},$$

$g \subseteq h$. So $h = f \cap^{\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})} (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B})$. □

Corollary 62. Let $\mathfrak{A}$, $\mathfrak{B}$ are sets of filters over boolean lattices. For every $f \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ and $\mathcal{A} \in \mathfrak{A}$ we have $f|_{\mathcal{A}} = f \cap (\mathcal{A} \times^{\mathsf{FCD}} 1^{\mathfrak{B}})$.

Proof. $f \cap (\mathcal{A} \times^{\mathsf{FCD}} 1^{\mathfrak{B}}) = I_{1^{\mathfrak{B}}}^{\mathsf{FCD}(\mathfrak{B})} \circ f \circ I_{\mathcal{A}}^{\mathsf{FCD}(\mathfrak{A})} = f \circ I_{\mathcal{A}}^{\mathsf{FCD}(\mathfrak{A})} = f|_{\mathcal{A}}$. □

Corollary 63. Let $\mathfrak{A}$, $\mathfrak{B}$ are sets of filters over boolean lattices. For every $f \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ and $\mathcal{A} \in \mathfrak{A}$, $\mathcal{B} \in \mathfrak{B}$ we have

$$f \cap^{\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})} (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) \neq 0^{\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})} \Leftrightarrow \mathcal{A} [f] \mathcal{B}.$$

Proof. $f \cap^{\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})} (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) \neq 0^{\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})} \Leftrightarrow \langle f \cap^{\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})} (\mathcal{A} \times^{\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})} \mathcal{B}) \rangle 1^{\mathfrak{A}} \neq 0^{\mathfrak{B}} \Leftrightarrow \left\langle I_{\mathcal{B}}^{\mathsf{FCD}(\mathfrak{B})} \circ f \circ I_{\mathcal{A}}^{\mathsf{FCD}(\mathfrak{A})} \right\rangle 1^{\mathfrak{A}} \neq 0^{\mathfrak{B}} \Leftrightarrow \left\langle I_{\mathcal{B}}^{\mathsf{FCD}(\mathfrak{B})} \right\rangle \langle f \rangle \left\langle I_{\mathcal{A}}^{\mathsf{FCD}(\mathfrak{A})} \right\rangle 1^{\mathfrak{A}} \neq 0^{\mathfrak{B}} \Leftrightarrow \mathcal{B} \cap^{\mathfrak{B}} \langle f \rangle (\mathcal{A} \cap^{\mathfrak{A}} 1^{\mathfrak{A}}) \neq 0^{\mathfrak{B}} \Leftrightarrow \mathcal{B} \cap^{\mathfrak{B}} \langle f \rangle \mathcal{A} \neq 0^{\mathfrak{B}} \Leftrightarrow \mathcal{A} [f] \mathcal{B}$. □

Theorem 64. Let $\mathfrak{A}$, $\mathfrak{B}$ are sets of filters over boolean lattices. Then the poset $\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ is separable.

Proof. Let $f, g \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ and $f \neq g$. By the theorem 12 $[f] \neq [g]$. That is there exist $x, y \in \mathfrak{A}$ such that $x [f] y \not\Leftrightarrow x [g] y$ that is $f \cap^{\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})} (x \times^{\mathsf{FCD}} y) \neq 0^{\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})} \not\Leftrightarrow g \cap^{\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})} (x \times^{\mathsf{FCD}} y) \neq 0^{\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})}$. Thus $\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ is separable. □

Theorem 65. Let $(\mathfrak{A}; \mathfrak{Z}_0)$ and $(\mathfrak{B}; \mathfrak{Z}_1)$ are primary filtrators over boolean lattices. If $S \in \mathscr{P}(\mathfrak{A} \times \mathfrak{B})$ then

$$\bigcap^{\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})} \{ \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \mid (\mathcal{A}; \mathcal{B}) \in S \} = \bigcap^{\mathfrak{A}} \operatorname{dom} S \times^{\mathsf{FCD}} \bigcap^{\mathfrak{B}} \operatorname{im} S.$$

Proof. If $x \in \operatorname{atoms}^{\mathfrak{A}}$ then by the theorem 56

$$\langle \bigcap^{\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})} \{ \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \mid (\mathcal{A}; \mathcal{B}) \in S \} \rangle x = \bigcap^{\mathfrak{B}} \{ \langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \rangle x \mid (\mathcal{A}; \mathcal{B}) \in S \}.$$

If $x \cap^{\mathfrak{A}} \bigcap^{\mathfrak{A}} \operatorname{dom} S \neq 0^{\mathfrak{A}}$ then

$$\forall (\mathcal{A}; \mathcal{B}) \in S : (x \cap^{\mathfrak{A}} \mathcal{A} \neq 0^{\mathfrak{A}} \wedge \langle \mathcal{A} \times^{\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})} \mathcal{B} \rangle x = \mathcal{B});$$
$$\{ \langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \rangle x \mid (\mathcal{A}; \mathcal{B}) \in S \} = \operatorname{im} S;$$

if $x \cap^{\mathfrak{A}} \bigcap^{\mathfrak{A}} \operatorname{dom} S = 0^{\mathfrak{A}}$ then

$$\exists (\mathcal{A}; \mathcal{B}) \in S : (x \cap^{\mathfrak{A}} \mathcal{A} = 0^{\mathfrak{A}} \wedge \langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \rangle x = 0^{\mathfrak{B}});$$
$$\{ \langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \rangle x \mid (\mathcal{A}; \mathcal{B}) \in S \} \ni 0^{\mathfrak{B}}.$$

So

$$\langle \bigcap^{\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})} \{ \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \mid (\mathcal{A}; \mathcal{B}) \in S \} \rangle x = \begin{cases} \bigcap^{\mathfrak{B}} \operatorname{im} S & \text{if } x \cap^{\mathfrak{A}} \bigcap^{\mathfrak{A}} \operatorname{dom} S \neq 0^{\mathfrak{A}}; \\ 0^{\mathfrak{B}} & \text{if } x \cap^{\mathfrak{A}} \bigcap^{\mathfrak{A}} \operatorname{dom} S = 0^{\mathfrak{A}}. \end{cases}$$

From this by corollary 53 (taking in account 47 in [3]) follows the statement of the theorem. □

Corollary 66. Let $(\mathfrak{A}; \mathfrak{Z}_0)$ and $(\mathfrak{B}; \mathfrak{Z}_1)$ are primary filtrators over boolean lattices.