From this $x\left[\bigcap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} R\right] y \Leftrightarrow \forall f \in R: x\,[f]\,y$.

1. From the former $y \in \operatorname{atoms}^{\mathfrak{B}} \langle \bigcap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} R\rangle x \Leftrightarrow y \cap^{\mathfrak{B}} \langle \bigcap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} R\rangle x \neq 0^{\mathfrak{B}} \Leftrightarrow \forall f \in R$: $y \cap^{\mathfrak{B}} \langle f\rangle x \neq 0^{\mathfrak{B}} \Leftrightarrow y \in \bigcap \langle \operatorname{atoms}^{\mathfrak{B}}\rangle\{\langle f\rangle x \mid f \in R\} \Leftrightarrow y \in \operatorname{atoms}^{\mathfrak{B}} \bigcap^{\mathfrak{B}}\{\langle f\rangle x \mid f \in R\}$ for every $y \in \operatorname{atoms}^{\mathfrak{B}}$.

$\mathfrak{B}$ is atomically separable by the corollary 17 in [3]. Thus

$$\langle \bigcap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} R\rangle x = \bigcap^{\mathfrak{B}}\{\langle f\rangle x \mid f \in R\}.$$

□

Theorem 57. Let $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ are posets of filter objects over some boolean lattices, $f \in \mathsf{FCD}(\mathfrak{A};\mathfrak{B})$, $g \in \mathsf{FCD}(\mathfrak{B};\mathfrak{C})$, $h \in \mathsf{FCD}(\mathfrak{A};\mathfrak{C})$. Then

$$g \circ f \not\asymp h \Leftrightarrow g \not\asymp h \circ f^{-1}.$$

Proof.

$$\begin{aligned}
g \circ f \not\asymp h \;&\Leftrightarrow\\
\exists a \in \operatorname{atoms} 1^{\mathfrak{A}}, c \in \operatorname{atoms} 1^{\mathfrak{C}}: a\,[(g \circ f) \cap h]\,c \;&\Leftrightarrow\\
\exists a \in \operatorname{atoms} 1^{\mathfrak{A}}, c \in \operatorname{atoms} 1^{\mathfrak{C}}: (a\,[g \circ f]\,c \wedge a\,[h]\,c) \;&\Leftrightarrow\\
\exists a \in \operatorname{atoms} 1^{\mathfrak{A}}, b \in \operatorname{atoms} 1^{\mathfrak{B}}, c \in \operatorname{atoms} 1^{\mathfrak{C}}: (a\,[f]\,b \wedge b\,[g]\,c \wedge a\,[h]\,c) \;&\Leftrightarrow\\
\exists b \in \operatorname{atoms} 1^{\mathfrak{B}}, c \in \operatorname{atoms} 1^{\mathfrak{C}}: (b\,[g]\,c \wedge b\,[h \circ f^{-1}]\,c) \;&\Leftrightarrow\\
\exists b \in \operatorname{atoms} 1^{\mathfrak{B}}, c \in \operatorname{atoms} 1^{\mathfrak{C}}: b\,[g \cap (h \circ f^{-1})]\,c \;&\Leftrightarrow\\
g \not\asymp h \circ f^{-1}.&
\end{aligned}$$

□

3.9 Direct product of elements

Definition 58. Let $\mathfrak{A}$ and $\mathfrak{B}$ are posets with least elements and $\mathcal{A} \in \mathfrak{A}$, $\mathcal{B} \in \mathfrak{B}$. *Funcoidal product* of $\mathcal{A}, \mathcal{B} \in \mathfrak{A}$ is such a pointfree funcoid $\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \in \mathsf{FCD}(\mathfrak{A};\mathfrak{B})$ that

$$\mathcal{X}\,[\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}]\,\mathcal{Y} \Leftrightarrow \mathcal{X} \not\asymp^{\mathfrak{A}} \mathcal{A} \wedge \mathcal{Y} \not\asymp^{\mathfrak{B}} \mathcal{B}.$$

Proposition 59. $\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}$ is really a pointfree funcoid and for every $\mathcal{X} \in \mathfrak{A}$

$$\langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}\rangle \mathcal{X} = \begin{cases} \mathcal{B} & \text{if } \mathcal{X} \not\asymp^{\mathfrak{A}} \mathcal{A};\\ 0^{\mathfrak{B}} & \text{if } \mathcal{X} \asymp^{\mathfrak{A}} \mathcal{A}.\end{cases}$$

Proof. Obvious. □

Proposition 60. Let $\mathfrak{A}$ and $\mathfrak{B}$ are bounded posets, $f \in \mathsf{FCD}(\mathfrak{A};\mathfrak{B})$, $\mathcal{A} \in \mathfrak{A}$, $\mathcal{B} \in \mathfrak{B}$. Then

$$f \subseteq \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \Leftrightarrow \operatorname{dom} f \subseteq \mathcal{A} \wedge \operatorname{im} f \subseteq \mathcal{B}.$$

Proof. If $f \subseteq \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}$ then $\operatorname{dom} f \subseteq \operatorname{dom}(\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) \subseteq \mathcal{A}$, $\operatorname{im} f \subseteq \operatorname{im}(\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) \subseteq \mathcal{B}$. If $\operatorname{dom} f \subseteq \mathcal{A} \wedge \operatorname{im} f \subseteq \mathcal{B}$ then

$$\forall \mathcal{X} \in \mathfrak{A}, \mathcal{Y} \in \mathfrak{B}: (\mathcal{X}\,[f]\,\mathcal{Y} \Rightarrow \mathcal{X} \not\asymp^{\mathfrak{A}} \mathcal{A} \wedge \mathcal{Y} \not\asymp^{\mathfrak{B}} \mathcal{B});$$

consequently $f \subseteq \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}$. □

Theorem 61. Let $\mathfrak{A}$, $\mathfrak{B}$ are sets of filters over boolean lattices. For every $f \in \mathsf{FCD}(\mathfrak{A};\mathfrak{B})$ and $\mathcal{A} \in \mathfrak{A}$, $\mathcal{B} \in \mathfrak{B}$

$$f \cap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) = I^{\mathsf{FCD}(\mathfrak{B})}_{\mathcal{B}} \circ f \circ I^{\mathsf{FCD}(\mathfrak{A})}_{\mathcal{A}}.$$

Proof. From above $\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$ is a (complete) lattice.