Let continue α' till a funcoid f (by the theorem 26): $\langle f\rangle\mathcal{X}=\bigcap^{\mathfrak{B}}\langle\alpha'\rangle\mathrm{up}^{(\mathfrak{A};\mathfrak{Z}_0)}\mathcal{X}$.

Let's prove the reverse of (5):

$$\begin{aligned}\bigcap^{\mathfrak{B}}\left\langle\bigcup^{\mathfrak{B}}\circ\langle\alpha\rangle\circ\mathrm{atoms}^{\mathfrak{A}}\right\rangle\mathrm{up}^{(\mathfrak{A};\mathfrak{Z}_0)}a &= \bigcap^{\mathfrak{B}}\left\langle\bigcup^{\mathfrak{B}}\circ\langle\alpha\rangle\right\rangle\langle\mathrm{atoms}^{\mathfrak{A}}\rangle\mathrm{up}^{(\mathfrak{A};\mathfrak{Z}_0)}a\\ &\subseteq \bigcap^{\mathfrak{B}}\left\langle\bigcup^{\mathfrak{B}}\circ\langle\alpha\rangle\right\rangle\{\{a\}\}\\ &= \bigcap^{\mathfrak{B}}\left\{\left(\bigcup^{\mathfrak{B}}\circ\langle\alpha\rangle\right)\{a\}\right\}\\ &= \bigcap^{\mathfrak{B}}\left\{\bigcup^{\mathfrak{B}}\langle\alpha\rangle\{a\}\right\}\\ &= \bigcap^{\mathfrak{B}}\left\{\bigcup^{\mathfrak{B}}\{\alpha a\}\right\}=\bigcap^{\mathfrak{B}}\{\alpha a\}=\alpha a.\end{aligned}$$

Finally,

$$\alpha a=\bigcap^{\mathfrak{B}}\left\langle\bigcup^{\mathfrak{B}}\circ\langle\alpha\rangle\circ\mathrm{atoms}^{\mathfrak{A}}\right\rangle\mathrm{up}\,a=\bigcap^{\mathfrak{B}}\langle\alpha'\rangle\mathrm{up}^{(\mathfrak{A};\mathfrak{Z}_0)}a=\langle f\rangle a,$$

so $\langle f\rangle$ is a continuation of α.

2. Consider the relation $\delta'\in\mathscr{P}(\mathfrak{Z}_0\times\mathfrak{Z}_1)$ defined by the formula (for every $X\in\mathfrak{Z}_0$, $Y\in\mathfrak{Z}_1$)

$$X\,\delta'\,Y\Leftrightarrow\exists x\in\mathrm{atoms}^{\mathfrak{A}}X,y\in\mathrm{atoms}^{\mathfrak{B}}Y:x\,\delta\,y.$$

Obviously $\neg(X\,\delta'\,0^{\mathfrak{Z}_1})$ and $\neg(0^{\mathfrak{Z}_0}\,\delta'\,Y)$.

$$\begin{aligned}(I\cup J)\,\delta'\,Y &\Leftrightarrow \exists x\in\mathrm{atoms}^{\mathfrak{A}}(I\cup J),y\in\mathrm{atoms}^{\mathfrak{B}}Y:x\,\delta\,y\\ &\Leftrightarrow \exists x\in\mathrm{atoms}^{\mathfrak{A}}I\cup\mathrm{atoms}^{\mathfrak{A}}J,y\in\mathrm{atoms}^{\mathfrak{B}}Y:x\,\delta\,y\\ &\Leftrightarrow \exists x\in\mathrm{atoms}^{\mathfrak{A}}I,y\in\mathrm{atoms}^{\mathfrak{B}}Y:x\,\delta\,y\vee\exists x\in\mathrm{atoms}^{\mathfrak{A}}J,y\in\mathrm{atoms}^{\mathfrak{B}}Y:x\,\delta\,y\\ &\Leftrightarrow I\,\delta'\,Y\vee J\,\delta'\,Y;\end{aligned}$$

similarly $X\,\delta'\,(I\cup J)\Leftrightarrow X\,\delta'\,I\vee X\,\delta'\,J$. Let's continue δ' till a funcoid f (by the theorem 26):

$$\mathcal{X}\,[f]\,\mathcal{Y}\Leftrightarrow\forall X\in\mathrm{up}^{(\mathfrak{A};\mathfrak{Z}_0)}\mathcal{X},Y\in\mathrm{up}^{(\mathfrak{B};\mathfrak{Z}_1)}\mathcal{Y}:X\,\delta'\,Y$$

The reverse of (7) implication is trivial, so

$$\forall X\in\mathrm{up}^{(\mathfrak{A};\mathfrak{Z}_0)}a,Y\in\mathrm{up}^{(\mathfrak{B};\mathfrak{Z}_1)}b\,\exists x\in\mathrm{atoms}^{\mathfrak{A}}X,y\in\mathrm{atoms}^{\mathfrak{B}}Y:x\,\delta\,y\Leftrightarrow a\,\delta\,b.$$

$\forall X\in\mathrm{up}^{(\mathfrak{A};\mathfrak{Z}_0)}a,Y\in\mathrm{up}^{(\mathfrak{B};\mathfrak{Z}_1)}b\,\exists x\in\mathrm{atoms}^{\mathfrak{A}}X,y\in\mathrm{atoms}^{\mathfrak{B}}Y:x\,\delta\,y\Leftrightarrow\forall X\in\mathrm{up}^{(\mathfrak{A};\mathfrak{Z}_0)}a,Y\in\mathrm{up}^{(\mathfrak{B};\mathfrak{Z}_1)}b$: $X\,\delta'\,Y\Leftrightarrow a\,[f]\,b$.

So $a\,\delta\,b\Leftrightarrow a\,[f]\,b$, that is $[f]$ is a continuation of δ. □

One of uses of the previous theorem is the proof of the following theorem:

Theorem 56. Let $(\mathfrak{A};\mathfrak{Z}_0)$ and $(\mathfrak{B};\mathfrak{Z}_1)$ are primary filtrators over boolean lattices. If $R\in\mathscr{P}\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$ and $x\in\mathrm{atoms}^{\mathfrak{A}}$, $y\in\mathrm{atoms}^{\mathfrak{B}}$, then

1. $\left\langle\bigcap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}R\right\rangle x=\bigcap^{\mathfrak{B}}\{\langle f\rangle x\mid f\in R\}$;
2. $x\left[\bigcap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}R\right]y\Leftrightarrow\forall f\in R:x\,[f]\,y$.

Proof.

2. Let denote $x\,\delta\,y\Leftrightarrow\forall f\in R:x\,[f]\,y$.

$$\begin{gathered}\forall X\in\mathrm{up}^{(\mathfrak{A};\mathfrak{Z}_0)}a,Y\in\mathrm{up}^{(\mathfrak{B};\mathfrak{Z}_1)}b\,\exists x\in\mathrm{atoms}^{\mathfrak{A}}X,y\in\mathrm{atoms}^{\mathfrak{B}}Y:x\,\delta\,y\Rightarrow\\ \forall f\in R,X\in\mathrm{up}^{(\mathfrak{A};\mathfrak{Z}_0)}a,Y\in\mathrm{up}^{(\mathfrak{B};\mathfrak{Z}_1)}b\,\exists x\in\mathrm{atoms}^{\mathfrak{A}}X,y\in\mathrm{atoms}^{\mathfrak{B}}Y:x\,[f]\,y\Rightarrow\\ \forall f\in R,X\in\mathrm{up}^{(\mathfrak{A};\mathfrak{Z}_0)}a,Y\in\mathrm{up}^{(\mathfrak{B};\mathfrak{Z}_1)}b:X\,[f]\,Y\Rightarrow\\ \forall f\in R:a\,[f]\,b\Leftrightarrow\\ a\,\delta\,b.\end{gathered}$$

So, by the theorem 55, δ can be continued till $[p]$ for some $p\in\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$.

For every $q\in\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$ such that $\forall f\in R:q\subseteq f$ we have $x\,[q]\,y\Rightarrow\forall f\in R:x\,[f]\,y\Leftrightarrow x\,\delta\,y\Leftrightarrow x\,[p]\,y$, so $q\subseteq p$. Consequently $p=\bigcap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}R$.