2. $(g \cup^{\mathsf{FCD}(\mathfrak{B};\mathfrak{C})} h) \circ f = g \circ f \cup^{\mathsf{FCD}(\mathfrak{A};\mathfrak{C})} h \circ f$ for $f \in \mathsf{FCD}(\mathfrak{A};\mathfrak{B})$ and $g, h \in \mathsf{FCD}(\mathfrak{B};\mathfrak{C})$.

**Proof.** I will prove only the first equality because the other is analogous.

For every $\mathcal{X} \in \mathfrak{A}$, $\mathcal{Y} \in \mathfrak{C}$

$$\begin{aligned}
\mathcal{X}\left[f \circ (g \cup^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} h)\right] \mathcal{Z} &\Leftrightarrow \exists y \in \operatorname{atoms}^{\mathfrak{B}}: (\mathcal{X}\left[g \cup^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} h\right] y \wedge y\,[f]\,\mathcal{Z}) \\
&\Leftrightarrow \exists y \in \operatorname{atoms}^{\mathfrak{B}}: ((\mathcal{X}\,[g]\,y \vee \mathcal{X}\,[h]\,y) \wedge y\,[f]\,\mathcal{Z}) \\
&\Leftrightarrow \exists y \in \operatorname{atoms}^{\mathfrak{B}}: ((\mathcal{X}\,[g]\,y \wedge y\,[f]\,\mathcal{Z}) \vee (\mathcal{X}\,[h]\,y \wedge y\,[f]\,\mathcal{Z})) \\
&\Leftrightarrow \exists y \in \operatorname{atoms}^{\mathfrak{B}}: (\mathcal{X}\,[g]\,y \wedge y\,[f]\,\mathcal{Z}) \vee \exists y \in \operatorname{atoms}^{\mathfrak{B}}: (\mathcal{X}\,[h]\,y \wedge y\,[f]\,\mathcal{Z}) \\
&\Leftrightarrow \mathcal{X}\,[f \circ g]\,\mathcal{Z} \vee \mathcal{X}\,[f \circ h]\,\mathcal{Z} \\
&\Leftrightarrow \mathcal{X}\left[f \circ g \cup^{\mathsf{FCD}(\mathfrak{A};\mathfrak{C})} f \circ h\right] \mathcal{Z}.
\end{aligned}$$

□

## 3.6 Domain and range of a pointfree funcoid

**Definition 38.** Let $\mathfrak{A}$ is a poset. The *identity pointfree funcoid* $I^{\mathsf{FCD}(\mathfrak{A})} = (\mathfrak{A}; \mathfrak{A}; (=)|_{\mathfrak{A}}; (=)|_{\mathfrak{A}})$.

It is trivial that identity funcoid is really a pointfree funcoid.

Let now $\mathfrak{A}$ is a meet-semilattice.

**Definition 39.** Let $a \in \mathfrak{A}$. The *restricted identity pointfree funcoid* $I_a^{\mathsf{FCD}(\mathfrak{A})} = (\mathfrak{A}; \mathfrak{A}; a \cap^{\mathfrak{A}}; a \cap^{\mathfrak{A}})$.

**Proposition 40.** The restricted pointfree funcoid is a pointfree funcoid.

**Proof.** We need to prove that $(a \cap^{\mathfrak{A}} x) \not\asymp^{\mathfrak{A}} y \Leftrightarrow (a \cap^{\mathfrak{A}} y) \not\asymp^{\mathfrak{A}} x$ what is obvious. □

**Obvious 41.** $\left(I_{\mathcal{A}}^{\mathsf{FCD}(\mathfrak{A})}\right)^{-1} = I_{\mathcal{A}}^{\mathsf{FCD}(\mathfrak{A})}$.

**Obvious 42.** $x\left[I_{\mathcal{A}}^{\mathsf{FCD}(\mathfrak{A})}\right] y \Leftrightarrow a \not\asymp^{\mathfrak{A}} x \cap^{\mathfrak{A}} y$ for every $x, y \in \mathfrak{A}$.

**Definition 43.** I will define *restricting* of a pointfree funcoid $f$ to an element $a \in \operatorname{Src} f$ by the formula $f|_a \overset{\text{def}}{=} f \circ I_a^{\mathsf{FCD}(\operatorname{Src} f)}$.

**Definition 44.** Let $f$ is a pointfree funcoid whose source has greatest element 1.

*Image* of $f$ will be defined by the formula $\operatorname{im} f = \langle f \rangle 1$.

**Definition 45.** *Domain* of a pointfree funcoid $f$ is defined by the formula $\operatorname{dom} f = \operatorname{im} f^{-1}$ (when $f$ has a poset with greatest element as its destination).

**Proposition 46.** $\langle f \rangle x = \langle f \rangle (x \cap^{\operatorname{Src} f} \operatorname{dom} f)$ for every pointfree funcoid $f$ whose destination is a separable poset with greatest element and source is a meet-semilattice and $x \in \operatorname{Src} f$.

**Proof.** For every $y \in \operatorname{Dst} f$ we have $y \not\asymp^{\operatorname{Dst} f} \langle f \rangle (x \cap^{\operatorname{Src} f} \operatorname{dom} f) \neq 0^{\operatorname{Dst} f} \Leftrightarrow x \cap^{\operatorname{Src} f} \operatorname{dom} f \cap^{\operatorname{Src} f} \langle f^{-1} \rangle y \neq 0^{\operatorname{Src} f} \Leftrightarrow x \cap^{\operatorname{Src} f} \operatorname{im} f^{-1} \cap^{\operatorname{Src} f} \langle f^{-1} \rangle y \neq 0^{\operatorname{Src} f} \Leftrightarrow x \cap^{\operatorname{Src} f} \langle f^{-1} \rangle y \neq 0^{\operatorname{Src} f} \Leftrightarrow y \not\asymp^{\operatorname{Dst} f} \langle f \rangle x$. Thus $\langle f \rangle x = \langle f \rangle (x \cap^{\operatorname{Src} f} \operatorname{dom} f)$ by separability of $\operatorname{Dst} f$. □

**Proposition 47.** $x \not\asymp^{\operatorname{Src} f} \operatorname{dom} f \Leftrightarrow (\langle f \rangle x$ is not least) for every pointfree funcoid $f$ whose destination is a poset with greatest element and $x \in \operatorname{Src} f$.

**Proof.** $x \not\asymp^{\operatorname{Src} f} \operatorname{dom} f \Leftrightarrow x \not\asymp^{\operatorname{Src} f} \langle f^{-1} \rangle 1 \Leftrightarrow 1^{\operatorname{Dst} f} \not\asymp^{\operatorname{Dst} f} \langle f \rangle x \Leftrightarrow (\langle f \rangle x$ is not least). □

**Corollary 48.** $\operatorname{dom} f = \bigcup^{\operatorname{Src} f} \{a \in \operatorname{atoms}^{\operatorname{Src} f} \mid \langle f \rangle a \neq 0^{\operatorname{Dst} f}\}$ for every pointfree funcoid $f$ whose destination is a bounded poset and source is an atomistic meet-semilattice.

**Proof.** For every $a \in \operatorname{atoms}^{\operatorname{Src} f}$ we have $a \not\asymp^{\operatorname{Src} f} \operatorname{dom} f \Leftrightarrow a \not\asymp^{\operatorname{Src} f} \operatorname{im} f^{-1} \Leftrightarrow a \not\asymp^{\operatorname{Src} f} \langle f^{-1} \rangle 1^{\operatorname{Dst} f} \Leftrightarrow 1^{\operatorname{Dst} f} \not\asymp \langle f \rangle a \Leftrightarrow \langle f \rangle a \neq 0^{\operatorname{Dst} f}$. So