**Proof.** $\mathfrak{A}$ and $\mathfrak{B}$ are separable accordingly obvious 20 in [3].

Then apply [1] taking in account the theorem 12. □

**Theorem 32.** Let $\mathfrak{A}$ and $\mathfrak{B}$ are starrish join-semilattices. Then:

1. $\left\langle f \cup^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} g \right\rangle x = \langle f \rangle x \cup^{\mathfrak{B}} \langle g \rangle x$ for every $x \in \mathfrak{A}$;
2. $\left[ f \cup^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} g \right] = [f] \cup [g]$.

**Proof.**

1. Let $\alpha \mathcal{X} \overset{\text{def}}{=} \langle f \rangle x \cup^{\mathfrak{B}} \langle g \rangle x$; $\beta \mathcal{Y} \overset{\text{def}}{=} \langle f^{-1} \rangle y \cup^{\mathfrak{A}} \langle g^{-1} \rangle y$ for every $x \in \mathfrak{A}$, $y \in \mathfrak{B}$. Then

$$\begin{aligned} y \not\asymp^{\mathfrak{B}} \alpha x &\Leftrightarrow y \not\asymp^{\mathfrak{B}} \langle f \rangle x \vee y \not\asymp^{\mathfrak{B}} \langle g \rangle x \\ &\Leftrightarrow x \not\asymp^{\mathfrak{A}} \langle f^{-1} \rangle y \vee x \not\asymp^{\mathfrak{A}} \langle g^{-1} \rangle y \\ &\Leftrightarrow x \not\asymp^{\mathfrak{A}} \langle f^{-1} \rangle y \cup^{\mathfrak{A}} \langle g^{-1} \rangle y \\ &\Leftrightarrow x \not\asymp^{\mathfrak{A}} \beta y. \end{aligned}$$

So $h = (\mathfrak{A}; \mathfrak{B}; \alpha; \beta)$ is a pointfree funcoid. Obviously $h \supseteq f$ and $h \supseteq g$. If $p \supseteq f$ and $p \supseteq g$ for some $p \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ then $\langle p \rangle x \supseteq \langle f \rangle x \cup^{\mathfrak{B}} \langle g \rangle x = \langle h \rangle x$ that is $p \supseteq h$. So $f \cup^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} g = h$.

2. $x \left[ f \cup^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} g \right] y \Leftrightarrow y \not\asymp^{\mathfrak{B}} \left\langle f \cup^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} g \right\rangle x \Leftrightarrow y \not\asymp^{\mathfrak{B}} \langle f \rangle x \cup^{\mathfrak{B}} \langle g \rangle x \Leftrightarrow y \not\asymp^{\mathfrak{B}} \langle f \rangle x \vee y \not\asymp^{\mathfrak{B}} \langle g \rangle x \Leftrightarrow x [f] y \vee x [g] y$ for every $x \in \mathfrak{A}$, $y \in \mathfrak{B}$. □

**Theorem 33.** Let $f$ is a pointfree funcoid from a separable poset $\mathfrak{A}$ to a separable poset $\mathfrak{B}$. If $\langle f \rangle$ is an injection, then $\langle f \rangle$ is an order embedding $\mathfrak{A} \rightarrow \mathfrak{B}$.

**Proof.** Suppose $x \supseteq y$ but $\langle f \rangle x \not\supseteq \langle f \rangle y$.

Then by separability of $\mathfrak{B}$ there exist $z \not\asymp \langle f \rangle y$ such that $z \asymp \langle f \rangle x$.

Thus $\langle f^{-1} \rangle z \asymp x$ and $\langle f^{-1} \rangle z \not\asymp y$ what is impossible for $x \supseteq y$. □

**Corollary 34.** Let $f$ is a pointfree funcoid from a separable poset $\mathfrak{A}$ to a separable poset $\mathfrak{B}$. If $\langle f \rangle$ is a bijection $\mathfrak{A} \rightarrow \mathfrak{B}$, then $\langle f \rangle$ is an order isomorphism $\mathfrak{A} \rightarrow \mathfrak{B}$.

## 3.5 More on composition of pointfree funcoids

**Proposition 35.** $[g \circ f] = [g] \circ \langle f \rangle = \langle g^{-1} \rangle^{-1} \circ [f]$ for every composable pointfree funcoids $f$ and $g$.

**Proof.** $x\ [g \circ f]\ y \Leftrightarrow y \not\asymp^{\operatorname{Dst} g} \langle g \circ f \rangle x \Leftrightarrow y \not\asymp^{\operatorname{Dst} g} \langle g \rangle \langle f \rangle x \Leftrightarrow \langle f \rangle x\ [g]\ y \Leftrightarrow x ([g] \circ \langle f \rangle) y$ for every $x \in \mathfrak{A}$, $y \in \mathfrak{B}$. Thus $[g \circ f] = [g] \circ \langle f \rangle$. $[g \circ f] = [(f^{-1} \circ g^{-1})^{-1}] = [f^{-1} \circ g^{-1}]^{-1} = ([f^{-1}] \circ \langle g^{-1} \rangle)^{-1} = \langle g^{-1} \rangle^{-1} \circ [f]$. □

**Theorem 36.** Let $f$ and $g$ are pointfree funcoids and $\mathfrak{A} = \operatorname{Dst} f = \operatorname{Src} g$ is an atomic poset. Then for every $\mathcal{X} \in \operatorname{Src} f$ and $\mathcal{Z} \in \operatorname{Dst} g$

$$\mathcal{X}\ [g \circ f]\ \mathcal{Z} \Leftrightarrow \exists y \in \operatorname{atoms}^{\mathfrak{A}} : (\mathcal{X}\ [f]\ y \wedge y\ [g]\ \mathcal{Z}).$$

**Proof.**

$$\begin{aligned} \exists y \in \operatorname{atoms}^{\mathfrak{A}} : (\mathcal{X}\ [f]\ y \wedge y\ [g]\ \mathcal{Z}) &\Leftrightarrow \exists y \in \operatorname{atoms}^{\mathfrak{A}} : (\mathcal{Z} \not\asymp^{\operatorname{Dst} g} \langle g \rangle y \wedge y \not\asymp^{\mathfrak{A}} \langle f \rangle \mathcal{X}) \\ &\Leftrightarrow \exists y \in \operatorname{atoms}^{\mathfrak{A}} : (\mathcal{Z} \not\asymp^{\operatorname{Dst} g} \langle g \rangle y \wedge y \subseteq \langle f \rangle \mathcal{X}) \\ &\Rightarrow \mathcal{Z} \not\asymp^{\operatorname{Dst} g} \langle g \rangle \langle f \rangle \mathcal{X} \\ &\Leftrightarrow \mathcal{X}\ [g \circ f]\ \mathcal{Z}. \end{aligned}$$

Reversely, if $\mathcal{X}\ [g \circ f]\ \mathcal{Z}$ then $\langle f \rangle \mathcal{X}\ [g]\ \mathcal{Z}$, consequently exists $y \in \operatorname{atoms}^{\mathfrak{A}} \langle f \rangle \mathcal{X}$ such that $y\ [g]\ \mathcal{Z}$; we have $\mathcal{X}\ [f]\ y$. □

**Theorem 37.** Let $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ are posets and $\mathfrak{B}$ is atomic. Then:

1. $f \circ (g \cup^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} h) = f \circ g \cup^{\mathsf{FCD}(\mathfrak{A};\mathfrak{C})} f \circ h$ for $g, h \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ and $f \in \mathsf{FCD}(\mathfrak{B}; \mathfrak{C})$.