Theorem 21. $(g\circ f)^{-1}=f^{-1}\circ g^{-1}$ for every composable pointfree funcoids f and g.

Proof.

$$\langle (g\circ f)^{-1}\rangle = \langle f^{-1}\rangle\circ\langle g^{-1}\rangle = \langle f^{-1}\circ g^{-1}\rangle;$$
$$\langle ((g\circ f)^{-1})^{-1}\rangle = \langle g\circ f\rangle = \langle (f^{-1}\circ g^{-1})^{-1}\rangle.$$

□

Proposition 22. $(h\circ g)\circ f=h\circ(g\circ f)$ for every composable pointfree funcoids f, g, h.

Proof.
$\langle (h\circ g)\circ f\rangle = \langle h\circ g\rangle\circ\langle f\rangle = \langle h\rangle\circ\langle g\rangle\circ\langle f\rangle = \langle h\rangle\circ\langle g\circ f\rangle = \langle h\circ(g\circ f)\rangle$.
$\langle ((h\circ g)\circ f)^{-1}\rangle = \langle f^{-1}\circ(h\circ g)^{-1}\rangle = \langle f^{-1}\circ g^{-1}\circ h^{-1}\rangle = \langle (g\circ f)^{-1}\circ h^{-1}\rangle = \langle (h\circ(g\circ f))^{-1}\rangle$. □

3.3 Pointfree funcoid as continuation

Proposition 23. Let f is a pointfree funcoid. Then for every $x\in\operatorname{Src} f$, $y\in\operatorname{Dst} f$ we have

1. If $(\operatorname{Src} f;\mathfrak{Z})$ is a filtrator with separable core then $x\,[f]\,y\Leftrightarrow\forall X\in\operatorname{up}^{(\operatorname{Src} f;\mathfrak{Z})}x: X\,[f]\,y$.
2. If $(\operatorname{Dst} f;\mathfrak{Z})$ is a filtrator with separable core then $x\,[f]\,y\Leftrightarrow\forall Y\in\operatorname{up}^{(\operatorname{Dst} f;\mathfrak{Z})}x: x\,[f]\,Y$.

Proof. We will prove only the second because the first is similar.
$x\,[f]\,y\Leftrightarrow y\not\asymp^{\operatorname{Dst} f}\langle f\rangle x\Leftrightarrow\forall Y\in\operatorname{up} y: Y\not\asymp^{\operatorname{Dst} f}\langle f\rangle x\Leftrightarrow\forall Y\in\operatorname{up} y: x\,[f]\,Y$. □

Corollary 24. Let f is a pointfree funcoid and $(\operatorname{Src} f;\mathfrak{Z}_0)$, $(\operatorname{Dst} f;\mathfrak{Z}_1)$ are filtrators with separable core. Then

$$x\,[f]\,y\Leftrightarrow\forall X\in\operatorname{up}^{(\operatorname{Src} f;\mathfrak{Z}_0)}x, Y\in\operatorname{up}^{(\operatorname{Dst} f;\mathfrak{Z}_1)}y: X\,[f]\,Y.$$

Proof. Apply the proposition twice. □

Theorem 25. Let f be a pointfree funcoid. Let $(\operatorname{Src} f;\mathfrak{Z}_0)$ is a finitely meet-closed filtrator with separable core and $(\operatorname{Dst} f;\mathfrak{Z}_1)$ is a primary filtrator over a distributive lattice.

$$\langle f\rangle x=\bigcap\nolimits^{\operatorname{Dst} f}\langle\langle f\rangle\rangle\operatorname{up}^{(\operatorname{Src} f;\mathfrak{Z})}x.$$

Proof. By the previous proposition for every $y\in\operatorname{Dst} f$:

$$y\not\asymp^{\operatorname{Dst} f}\langle f\rangle x\Leftrightarrow x\,[f]\,y\Leftrightarrow\forall X\in\operatorname{up}^{(\operatorname{Src} f;\mathfrak{Z}_0)}x: X\,[f]\,y\Leftrightarrow\forall X\in\operatorname{up}^{(\operatorname{Src} f;\mathfrak{Z}_0)}x: y\not\asymp^{\operatorname{Dst} f}\langle f\rangle X.$$

Let's denote $W=\left\{y\cap^{\operatorname{Dst} f}\langle f\rangle X\mid X\in\operatorname{up}^{(\operatorname{Src} f;\mathfrak{Z}_0)}x\right\}$. We will prove that W is a generalized filter base over $\mathfrak{Z}_1$. To prove this enough to show that $V=\left\{\langle f\rangle X\mid X\in\operatorname{up}^{(\operatorname{Src} f;\mathfrak{Z}_0)}x\right\}$ is a generalized filter base.

Let $\mathcal{P},\mathcal{Q}\in V$. Then $\mathcal{P}=\langle f\rangle A$, $\mathcal{Q}=\langle f\rangle B$ where $A,B\in\operatorname{up}^{(\operatorname{Src} f;\mathfrak{Z}_0)}x$; $A\cap^{\mathfrak{Z}_0}B\in\operatorname{up}^{(\operatorname{Src} f;\mathfrak{Z}_0)}x$ (used the fact that it is a finitely meet-closed and the theorem 29 in [3]) and $\mathcal{R}\subseteq\mathcal{P}\cap^{\operatorname{Dst} f}\mathcal{Q}$ for $\mathcal{R}=\langle f\rangle(A\cap^{\mathfrak{Z}_0}B)\in V$. So V is a generalized filter base and thus W is a generalized filter base.

$0^{\operatorname{Dst} f}\notin W\Leftrightarrow\bigcap^{\operatorname{Dst} f}W\neq 0^{\operatorname{Dst} f}$ by the properties of generalized filter bases. That is

$$\forall X\in\operatorname{up}^{(\operatorname{Src} f;\mathfrak{Z}_0)}x: y\cap^{\operatorname{Dst} f}\langle f\rangle X\neq 0^{\operatorname{Dst} f}\Leftrightarrow y\cap^{\operatorname{Dst} f}\bigcap\nolimits^{\operatorname{Dst} f}\langle\langle f\rangle\rangle\operatorname{up}^{(\operatorname{Src} f;\mathfrak{Z}_0)}x\neq 0^{\operatorname{Dst} f}.$$

Comparing with the above, $y\cap^{\operatorname{Dst} f}\langle f\rangle x\neq 0^{\operatorname{Dst} f}\Leftrightarrow y\cap^{\operatorname{Dst} f}\bigcap^{\operatorname{Dst} f}\langle\langle f\rangle\rangle\operatorname{up}^{(\operatorname{Src} f;\mathfrak{Z}_0)}x\neq 0^{\operatorname{Dst} f}$. So $\langle f\rangle x=\bigcap^{\operatorname{Dst} f}\langle\langle f\rangle\rangle\operatorname{up}^{(\operatorname{Src} f;\mathfrak{Z}_0)}x$ because $\operatorname{Dst} f$ is separable. □

Theorem 26. Let $(\mathfrak{A};\mathfrak{Z}_0)$ and $(\mathfrak{B};\mathfrak{Z}_1)$ are primary filtrators over boolean lattices.

1. A function $\alpha\in\mathfrak{B}^{\mathfrak{Z}_0}$ conforming to the formulas (for every $I,J\in\mathfrak{Z}_0$)

$$\alpha 0^{\mathfrak{Z}_0}=0^{\mathfrak{B}},\quad \alpha(I\cup^{\mathfrak{Z}_0}J)=\alpha I\cup^{\mathfrak{B}}\alpha J$$