Proof. $y \not\asymp \langle f\rangle 0^{\mathrm{Src}\, f} \Leftrightarrow 0^{\mathrm{Src}\, f} \not\asymp \langle f^{-1}\rangle y \Leftrightarrow 0 \Leftrightarrow y \not\asymp 0^{\mathrm{Dst}\, f}$. Thus by separability, $\langle f\rangle 0^{\mathrm{Src}\, f} = 0^{\mathrm{Dst}\, f}$. □

Proposition 14. If $\mathrm{Dst}\, f$ is a separable meet semilattice with least element then $\langle f\rangle$ is a monotone function (for a pointfree funcoid f).

Proof. $a \subseteq b \Rightarrow \forall x \in \mathrm{Dst}\, f: (a \not\asymp \langle f^{-1}\rangle x \Rightarrow b \not\asymp \langle f^{-1}\rangle x) \Rightarrow \forall x \in \mathfrak{A}: (x \not\asymp \langle f\rangle a \Rightarrow x \not\asymp \langle f\rangle b) \Rightarrow \langle f\rangle a \subseteq \langle f\rangle b$ (used the theorem 19 in [3]). □

Theorem 15. Let f is a pointfree funcoid from a distributive lattice $\mathrm{Src}\, f$ with least element to a separable distributive lattice $\mathrm{Dst}\, f$ with least element. Then $\langle f\rangle(i \cup^{\mathrm{Src}\, f} j) = \langle f\rangle i \cup^{\mathrm{Dst}\, f} \langle f\rangle j$ for every $i, j \in \mathrm{Src}\, f$. [TODO: In the book this theorem is strenghtened for starrish semilattices instead of distributive lattices.]

Proof.

$$\begin{aligned}
\star\langle f\rangle(i \cup^{\mathrm{Src}\, f} j) &= \\
\{y \in \mathfrak{A} \mid y \cap^{\mathrm{Dst}\, f} \langle f\rangle(i \cup^{\mathrm{Src}\, f} j) \neq 0^{\mathrm{Dst}\, f}\} &= \\
\{y \in \mathfrak{A} \mid (i \cup^{\mathrm{Src}\, f} j) \cap^{\mathrm{Src}\, f} \langle f^{-1}\rangle y \neq 0^{\mathrm{Src}\, f}\} &= \\
\{y \in \mathfrak{A} \mid (i \cap^{\mathrm{Src}\, f} \langle f^{-1}\rangle y) \cup^{\mathrm{Src}\, f} (j \cap^{\mathrm{Src}\, f} \langle f^{-1}\rangle y) \neq 0^{\mathrm{Src}\, f}\} &= \\
\{y \in \mathfrak{A} \mid i \cap^{\mathrm{Src}\, f} \langle f^{-1}\rangle y \neq 0^{\mathrm{Src}\, f} \vee j \cap^{\mathrm{Src}\, f} \langle f^{-1}\rangle y \neq 0^{\mathrm{Src}\, f}\} &= \\
\{y \in \mathfrak{A} \mid y \cap^{\mathrm{Dst}\, f} \langle f\rangle i \neq 0^{\mathrm{Dst}\, f} \vee y \cap^{\mathrm{Dst}\, f} \langle f\rangle j \neq 0^{\mathrm{Dst}\, f}\} &= \\
\{y \in \mathfrak{A} \mid (y \cap^{\mathrm{Dst}\, f} \langle f\rangle i) \cup^{\mathrm{Dst}\, f} (y \cap^{\mathrm{Dst}\, f} \langle f\rangle j) \neq 0^{\mathrm{Dst}\, f}\} &= \\
\{y \in \mathfrak{A} \mid y \cap^{\mathrm{Dst}\, f} (\langle f\rangle i \cup^{\mathrm{Dst}\, f} \langle f\rangle j) \neq 0^{\mathrm{Dst}\, f}\} &= \\
\star(\langle f\rangle i \cup^{\mathrm{Dst}\, f} \langle f\rangle j). &
\end{aligned}$$

Thus $\langle f\rangle(i \cup^{\mathrm{Src}\, f} j) = \langle f\rangle i \cup^{\mathrm{Src}\, f} \langle f\rangle j$ by separability. □

Proposition 16. Let f is a pointfree funcoid. Then: [TODO: This theorem is also strenghtened in the book.]

1. $k\, [f]\, i \cup j \Leftrightarrow k\, [f]\, i \vee k\, [f]\, k$ for every $i, j \in \mathrm{Dst}\, f$, $k \in \mathrm{Src}\, f$ if $\mathrm{Dst}\, f$ is a distributive lattice with least element.
2. $i \cup j\, [f]\, k \Leftrightarrow i\, [f]\, k \vee j\, [f]\, k$ for every $i, j \in \mathrm{Src}\, f$, $k \in \mathrm{Dst}\, f$ if $\mathrm{Src}\, f$ is a distributive lattice with least element.

Proof. 1. $k\, [f]\, i \cup^{\mathrm{Dst}\, f} j \Leftrightarrow (i \cup j) \cap^{\mathrm{Dst}\, f} \langle f\rangle k \neq 0^{\mathrm{Dst}\, f} \Leftrightarrow (i \cap^{\mathrm{Dst}\, f} \langle f\rangle k) \cup (j \cap^{\mathrm{Dst}\, f} \langle f\rangle k) \neq 0^{\mathrm{Dst}\, f} \Leftrightarrow i \cap^{\mathrm{Dst}\, f} \langle f\rangle k \neq 0^{\mathrm{Dst}\, f} \vee j \cap^{\mathrm{Dst}\, f} \langle f\rangle k \neq 0^{\mathrm{Dst}\, f} \Leftrightarrow k\, [f]\, i \vee k\, [f]\, j$.

2. Similar. □

3.2 Composition of pointfree funcoids

Definition 17. *Composition* of pointfree funcoids is defined by the formula

$$(\mathfrak{B}; \mathfrak{C}; \alpha_2; \beta_2) \circ (\mathfrak{A}; \mathfrak{B}; \alpha_1; \beta_1) = (\mathfrak{A}; \mathfrak{C}; \alpha_2 \circ \alpha_1; \beta_1 \circ \beta_2).$$

Definition 18. I will call funcoids f and g *composable* when $\mathrm{Dst}\, f = \mathrm{Src}\, g$.

Proposition 19. If f, g are pointfree funcoids and $\mathrm{Dst}\, f = \mathrm{Src}\, g$ then $g \circ f$ is pointfree funcoid.

Proof. Let $f = (\mathfrak{A}; \mathfrak{B}; \alpha_1; \beta_1)$, $g = (\mathfrak{B}; \mathfrak{C}; \alpha_2; \beta_2)$. For every $x, y \in \mathfrak{A}$ we have

$$y \not\asymp^{\mathfrak{C}} (\alpha_2 \circ \alpha_1) x \Leftrightarrow y \not\asymp^{\mathfrak{C}} \alpha_2 \alpha_1 x \Leftrightarrow \alpha_1 x \not\asymp^{\mathfrak{B}} \beta_2 y \Leftrightarrow x \not\asymp^{\mathfrak{A}} \beta_1 \beta_2 y \Leftrightarrow x \not\asymp^{\mathfrak{A}} (\beta_1 \circ \beta_2) y.$$

So $(\mathfrak{A}; \mathfrak{C}; \alpha_2 \circ \alpha_1; \beta_1 \circ \beta_2)$ is a pointfree funcoid. □

Obvious 20. $\langle g \circ f\rangle = \langle g\rangle \circ \langle f\rangle$ for every composable pointfree funcoids f and g.