In addition to $\operatorname{atoms}^{\mathfrak{A}} x$, the set of atoms under an element x of a poset $\mathfrak{A}$ we will write just $\operatorname{atoms}^{\mathfrak{A}}$ for all atoms of a poset $\mathfrak{A}$.

3 Pointfree funcoids

3.1 Definition

Definition 1. *Pointfree funcoid* is a quadruple $(\mathfrak{A}; \mathfrak{B}; \alpha; \beta)$ where $\mathfrak{A}$ and $\mathfrak{B}$ are posets, $\alpha \in \mathfrak{B}^{\mathfrak{A}}$ and $\beta \in \mathfrak{A}^{\mathfrak{B}}$ such that

$$\forall x \in \mathfrak{A}, y \in \mathfrak{B}: (y \not\asymp^{\mathfrak{B}} \alpha x \Leftrightarrow x \not\asymp^{\mathfrak{A}} \beta y).$$

Definition 2. The *source* $\operatorname{Src}(\mathfrak{A}; \mathfrak{B}; \alpha; \beta) = \mathfrak{A}$ and *destination* $\operatorname{Dst}(\mathfrak{A}; \mathfrak{B}; \alpha; \beta) = \mathfrak{B}$ for every pointfree funcoid $(\mathfrak{A}; \mathfrak{B}; \alpha; \beta)$.

Definition 3. I will denote $\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ the set of pointfree funcoids from $\mathfrak{A}$ to $\mathfrak{B}$ (that is with source $\mathfrak{A}$ and destination $\mathfrak{B}$), for every posets $\mathfrak{A}$ and $\mathfrak{B}$.

Proposition 4. If $\mathfrak{A}$ and $\mathfrak{B}$ have least elements, then $\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ has least element.

Proof. It is $(\mathfrak{A}; \mathfrak{B}; \mathfrak{A} \times \{0^{\mathfrak{B}}\}; \mathfrak{B} \times \{0^{\mathfrak{A}}\})$. □

Definition 5. $\langle(\mathfrak{A}; \mathfrak{B}; \alpha; \beta)\rangle \overset{\text{def}}{=} \alpha$ for a pointfree funcoid $(\mathfrak{A}; \mathfrak{B}; \alpha; \beta)$.

Definition 6. $(\mathfrak{A}; \mathfrak{B}; \alpha; \beta)^{-1} = (\mathfrak{B}; \mathfrak{A}; \beta; \alpha)$ for a pointfree funcoid $(\mathfrak{A}; \mathfrak{B}; \alpha; \beta)$.

Proposition 7. If f is a pointfree funcoid then f^{-1} is also a pointfree funcoid.

Proof. Follows from symmetry in the definition of pointfree funcoid. □

Obvious 8. $(f^{-1})^{-1} = f$ for a pointfree funcoid f.

Definition 9. The relation $[f] \in \mathscr{P}(\operatorname{Src} f \times \operatorname{Dst} f)$ is defined by the formula (for every pointfree funcoid f and $x \in \operatorname{Src} f$, $y \in \operatorname{Dst} f$)

$$x\,[f]\,y \overset{\text{def}}{=} y \not\asymp^{\operatorname{Dst} f} \langle f\rangle x.$$

Obvious 10. $x\,[f]\,y \Leftrightarrow y \not\asymp^{\operatorname{Dst} f} \langle f\rangle x \Leftrightarrow x \not\asymp^{\operatorname{Src} f} \langle f^{-1}\rangle y$ for every pointfree funcoid f and $x \in \operatorname{Src} f$, $y \in \operatorname{Dst} f$.

Obvious 11. $[f^{-1}] = [f]^{-1}$ for a pointfree funcoid f.

Theorem 12. Let $\mathfrak{A}$ and $\mathfrak{B}$ are posets. Then:

1. If $\mathfrak{A}$ is separable, for given value of $\langle f\rangle$ exists no more than one $f \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$;
2. If $\mathfrak{A}$ and $\mathfrak{B}$ are separable, for given value of $[f]$ exists no more than one $f \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$.

Proof. Let $f, g \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$.

1. Let $\langle f\rangle = \langle g\rangle$. Then for every $x \in \mathfrak{A}$, $y \in \mathfrak{B}$ we have $x \not\asymp^{\mathfrak{A}} \langle f^{-1}\rangle y \Leftrightarrow y \not\asymp^{\mathfrak{B}} \langle f\rangle x \Leftrightarrow y \not\asymp^{\mathfrak{B}} \langle g\rangle x \Leftrightarrow x \not\asymp^{\mathfrak{A}} \langle g^{-1}\rangle y$ and thus by separability of $\mathfrak{A}$ we have $\langle f^{-1}\rangle y = \langle g^{-1}\rangle y$ that is $\langle f^{-1}\rangle = \langle g^{-1}\rangle$ and so $f = g$.
2. Let $[f] = [g]$. Then for every $x \in \mathfrak{A}$, $y \in \mathfrak{B}$ we have $y \not\asymp^{\mathfrak{B}} \langle f\rangle x \Leftrightarrow x\,[f]\,y \Leftrightarrow x\,[g]\,y \Leftrightarrow y \not\asymp^{\mathfrak{B}} \langle g\rangle x$ and thus by separability of $\mathfrak{B}$ we have $\langle f\rangle x = \langle g\rangle x$ that is $\langle f\rangle = \langle g\rangle$. Similarly we have $\langle f^{-1}\rangle = \langle g^{-1}\rangle$. Thus $f = g$. □

Proposition 13. If $\operatorname{Dst} f$ is separable, then $\langle f\rangle 0^{\operatorname{Src} f} = 0^{\operatorname{Dst} f}$ for every pointfree funcoid f.