$\forall X \in a, Y \in b \forall i \in \operatorname{dom} f : \Pr_i X [f_i]^* \Pr_i Y$.
Then $\forall X \in \langle \Pr_i \rangle a, Y \in \langle \Pr_i \rangle b : X [f_i]^* Y$.
Thus by the lemma $\forall X \in \prod \langle \uparrow^{\operatorname{Src} f_i} \rangle \langle \Pr_i \rangle a, Y \in \prod \langle \uparrow^{\operatorname{Dst} f_i} \rangle \langle \Pr_i \rangle b : X [f_i]^* Y$.
$\forall X \in \Pr_i a, Y \in \Pr_i b : X [f_i]^* Y$.
Thus $\Pr_i a [f_i] \Pr_i b$. So $\forall i \in \operatorname{dom} f : \Pr_i a [f_i] \Pr_i b$ and thus $a \left[f \times^{(A)} g \right] b$. □

Remark 171. It seems that the proof of the above theorem can be simplified using cross-composition product.

Theorem 172. $\prod^{(A)}_{i \in n} (g_i \circ f_i) = \prod^{(A)} g \circ \prod^{(A)} f$ for indexed (by an index set n) families f and g of funcoids such that $\forall i \in n : \operatorname{Dst} f_i = \operatorname{Src} g_i$.

Proof. Let a, b are ultrafilters on $\prod_{i \in n} \operatorname{Src} f_i$ and $\prod_{i \in n} \operatorname{Dst} g_i$ correspondingly, $a \left[\prod^{(A)}_{i \in n} (g_i \circ f_i) \right] b \Leftrightarrow \forall i \in \operatorname{dom} f : \Pr_i a [g_i \circ f_i] \Pr_i b \Leftrightarrow \forall i \in \operatorname{dom} f \exists C \in \operatorname{atoms}^{\mathfrak{F}^{\prod_{i \in n} \operatorname{Dst} f_i}} : (\Pr_i a [f_i] C \wedge C [g_i] \Pr_i b) \Leftrightarrow \forall i \in \operatorname{dom} f \exists c \in \operatorname{atoms}^{\mathsf{RLD}(\lambda i \in n : \operatorname{Dst} f)} : (\Pr_i a [f_i] \Pr_i c \wedge \Pr_i c [g_i] \Pr_i b) \Leftarrow \exists c \in \operatorname{atoms}^{\mathsf{RLD}(\lambda i \in n : \operatorname{Dst} f)} \forall i \in \operatorname{dom} f : (\Pr_i a [f_i] \Pr_i c \wedge \Pr_i c [g_i] \Pr_i b) \Leftrightarrow \exists c \in \operatorname{atoms}^{\mathsf{RLD}(\lambda i \in n : \operatorname{Dst} f)} : \left(a \left[\prod^{(A)} f \right] c \wedge c \left[\prod^{(A)} g \right] b \right) \Leftrightarrow a \left[\prod^{(A)} g \circ \prod^{(A)} f \right] b$.

Let

$$\forall i \in \operatorname{dom} f \exists c \in \operatorname{atoms}^{\mathsf{RLD}(\lambda i \in n : \operatorname{Dst} f)} : (\Pr_i a [f_i] \Pr_i c \wedge \Pr_i c [g_i] \Pr_i b).$$

Then there exists $c' \in \operatorname{atoms}^{\mathsf{RLD}(\lambda i \in n : \operatorname{Dst} f)}$ such that

$$\forall i \in \operatorname{dom} f : (\Pr_i a [f_i] \Pr_i c'_i \wedge \Pr_i c'_i [g_i] \Pr_i b).$$

Then take $c'' = \prod^{\mathsf{RLD}} c'$. Then $\forall i \in \operatorname{dom} f : (\Pr_i a [f_i] \Pr_i c''_i \wedge \Pr_i c''_i [g_i] \Pr_i b)$. Thus

$$\exists c \in \operatorname{atoms}^{\mathsf{RLD}(\lambda i \in n : \operatorname{Dst} f)} \forall i \in \operatorname{dom} f : (\Pr_i a [f_i] \Pr_i c \wedge \Pr_i c [g_i] \Pr_i b).$$

We have $a \left[\prod^{(A)}_{i \in n} (g_i \circ f_i) \right] b \Leftrightarrow a \left[\prod^{(A)} g \circ \prod^{(A)} f \right] b$. □

Proposition 173. $\prod^{\mathsf{RLD}} a \left[\prod^{(A)} f \right] \prod^{\mathsf{RLD}} b \Leftrightarrow \forall i \in \operatorname{dom} f : a_i [f_i] b_i$ for an indexed family f of funcoids and indexed families a abd b of filters where $a_i \in \mathfrak{F}(\operatorname{Src} f)$, $b_i \in \mathfrak{F}(\operatorname{Dst} f)$ for every $i \in \operatorname{dom} f$.

Proof. $\prod^{\mathsf{RLD}} a \left[\prod^{(A)} f \right] \prod^{\mathsf{RLD}} b \Leftrightarrow \exists x \in \operatorname{atoms} \prod^{\mathsf{RLD}} a, y \in \operatorname{atoms} \prod^{\mathsf{RLD}} b : x \left[\prod^{(A)} f \right] y \Leftrightarrow \exists x \in \operatorname{atoms} \prod^{\mathsf{RLD}} a, y \in \operatorname{atoms} \prod^{\mathsf{RLD}} b \forall i \in \operatorname{dom} f : \Pr_i x [f_i] \Pr_i y \Leftrightarrow \exists x \in \operatorname{atoms} \prod^{\mathsf{RLD}} a, y \in \operatorname{atoms} \prod^{\mathsf{RLD}} b \forall i \in \operatorname{dom} f : a_i [f_i] b_i \Leftrightarrow \forall i \in \operatorname{dom} f : a_i [f_i] b_i$. □

15 On products and projections

Conjecture 174. For discrete funcoids $\prod^{(C)}$ and $\prod^{(A)}$ coincide with the conventional product of binary relations.

15.1 Staroidal product

Let f is a staroid components of whose form are boolean lattices.

Definition 175. *Staroidal projection* of a staroid

$$\Pr^{\mathsf{Strd}}_k f = \langle f \rangle_k \left(\lambda i \in (\operatorname{arity} f) \setminus \{k\} : 1^{(\operatorname{form} f)_i} \right).$$