Proof. It follows from the theorem ?? in [3]. □

Conjecture 148. $\operatorname{GR}\operatorname{StarComp}(a \sqcup^{\mathsf{pFCD}} b; f) = \operatorname{GR}\operatorname{StarComp}(a; f) \sqcup^{\mathsf{pFCD}} \operatorname{GR}\operatorname{StarComp}(b; f)$ if f is a pointfree funcoid and a, b are multifuncoids of the same form, composable with f.

12 More on cross-composition of funcoids

Lemma 149. Let f is a staroid such that $(\operatorname{form} f)_i$ is a boolean lattice for each $i \in \operatorname{arity} f$. Let $a \in \prod_{i \in \operatorname{arity} f} \mathfrak{F}^{(\operatorname{form} f)_i}$.

If $\upuparrows f \sqsubseteq \prod^{\mathsf{Strd}} a$ then $\upuparrows f = \operatorname{StarComp}\left(\upuparrows f; \lambda i \in \operatorname{dom} \mathfrak{A}: I_{a_i}^{\mathsf{FCD}(\mathfrak{A}_i)} \right)$.

Proof. Let $\upuparrows f \sqsubseteq \prod^{\mathsf{Strd}} a$. Then $L \in \operatorname{GR} \upuparrows f \Rightarrow L \not\asymp a$.

$L \in \operatorname{GR}\operatorname{StarComp}\left(\upuparrows f; \lambda i \in \operatorname{dom} \mathfrak{A}: I_{a_i}^{\mathsf{FCD}(\mathfrak{A}_i)} \right) \Leftrightarrow \exists y \in \operatorname{GR} \upuparrows f \cap \prod_{i \in n} \operatorname{atoms} \mathfrak{A}_i \forall i \in n: y_i \left[I_{a_i}^{\mathsf{FCD}(\mathfrak{A}_i)} \right] L_i \Leftrightarrow \exists y \in \operatorname{GR} \upuparrows f \cap \prod_{i \in n} \operatorname{atoms} \mathfrak{A}_i \forall i \in n: (y_i \sqsubseteq L_i \wedge y_i \sqsubseteq a_i) \Leftrightarrow \exists y \in \operatorname{GR} \upuparrows f \cap \prod_{i \in n} \operatorname{atoms} \mathfrak{A}_i \forall i \in n: (y_i \not\asymp L_i \wedge y_i \not\asymp a_i) \Leftrightarrow \exists y \in \operatorname{GR} \upuparrows f \cap \prod_{i \in n} \operatorname{atoms} \mathfrak{A}_i \forall i \in n: y_i \not\asymp L_i$ because $\upuparrows f \in \operatorname{GR} g \Rightarrow y \not\asymp a$.

If $L \in \upuparrows f$ then there exists $y \in \operatorname{GR} \upuparrows f \cap \prod_{i \in n} \operatorname{atoms} \mathfrak{A}_i$ such as $y \sqsubseteq L$ and thus $\forall i \in n: y_i \not\asymp L_i$ (by the theorem 81).

We have $L \in \operatorname{GR}\operatorname{StarComp}\left(\upuparrows f; \lambda i \in \operatorname{dom} \mathfrak{A}: I_{a_i}^{\mathsf{FCD}(\mathfrak{A}_i)} \right) \Leftarrow L \in \upuparrows f$ that is $\operatorname{GR}\operatorname{StarComp}\left(\upuparrows f; \lambda i \in \operatorname{dom} \mathfrak{A}: I_{a_i}^{\mathsf{FCD}(\mathfrak{A}_i)} \right) \sqsupseteq \upuparrows f$. The other directoin is obvious. □

Theorem 150. Let f is a staroid such that $(\operatorname{form} f)_i$ is a boolean lattice for each $i \in \operatorname{arity} f$. Let $a \in \prod_{i \in \operatorname{arity} f} \mathfrak{F}^{(\operatorname{form} f)_i}$. Then

$$\upuparrows f \sqcap^{\mathsf{FCD}(\operatorname{form} f)} \prod^{\mathsf{Strd}} a = \operatorname{StarComp}\left(\upuparrows f; \lambda i \in \operatorname{dom} \mathfrak{A}: I_{a_i}^{\mathsf{FCD}(\mathfrak{A}_i)} \right).$$

Proof. $h \overset{\text{def}}{=} \operatorname{StarComp}\left(\upuparrows f; \lambda i \in \operatorname{dom} \mathfrak{A}: I_{a_i}^{\mathsf{FCD}(\mathfrak{A}_i)} \right)$.

Obviously $h \sqsubseteq \upuparrows f$ and $h \sqsubseteq \prod^{\mathsf{Strd}} a$.

Suppose $g \sqsubseteq \upuparrows f$ and $g \sqsubseteq \prod^{\mathsf{Strd}} a$.

$x \in g \Leftrightarrow x \in \operatorname{StarComp}\left(g; \lambda i \in \operatorname{dom} \mathfrak{A}: I_{a_i}^{\mathsf{FCD}(\mathfrak{A}_i)} \right) \Rightarrow x \in \operatorname{StarComp}\left(f; \lambda i \in \operatorname{dom} \mathfrak{A}: I_{a_i}^{\mathsf{FCD}(\mathfrak{A}_i)} \right) \Leftrightarrow x \in h$ (used the proposition above).

So $g \sqsubseteq h$. □

Corollary 151. Let f is a completary staroid such that $(\operatorname{form} f)_i$ is a boolean lattice for each $i \in \operatorname{arity} f$. Let $a \in \prod_{i \in \operatorname{arity} f} \mathfrak{F}^{(\operatorname{form} f)_i}$. Then

$$\upuparrows f \sqcap^{\mathsf{cStrd}(\operatorname{form} f)} \prod^{\mathsf{Strd}} a = \operatorname{StarComp}\left(\upuparrows f; \lambda i \in \operatorname{dom} \mathfrak{A}: I_{a_i}^{\mathsf{FCD}(\mathfrak{A}_i)} \right).$$

Proof. Using the theorem 140. □

Theorem 152. Let f is a staroid such that $(\operatorname{form} f)_i$ is a boolean lattice for each $i \in \operatorname{arity} f$. Let $a \in \prod_{i \in \operatorname{arity} f} \mathfrak{F}^{(\operatorname{form} f)_i}$. Then $\upuparrows f \not\asymp^{\mathsf{FCD}(\operatorname{form} f)} \prod^{\mathsf{Strd}} a \Leftrightarrow a \in \upuparrows f$.

Proof. $\upuparrows f \not\asymp^{\mathsf{FCD}(\operatorname{form} f)} \prod^{\mathsf{Strd}} a \Leftrightarrow \upuparrows f \sqcap^{\mathsf{FCD}(\operatorname{form} f)} \prod^{\mathsf{Strd}} a \neq 0 \Leftrightarrow \operatorname{StarComp}\left(\upuparrows f; \lambda i \in \operatorname{arity} f: I_{a_i}^{\mathsf{FCD}(\mathfrak{A}_i)} \right) \neq 0^{\mathsf{FCD}(\operatorname{form} f)} \Leftrightarrow \exists L \in \mho^n, y \in \operatorname{GR} \upuparrows f \cap \prod_{i \in n} \operatorname{atoms} \mathfrak{A}_i \forall i \in n: y_i \left[I_{a_i}^{\mathsf{FCD}(\mathfrak{A}_i)} \right] L_i \Leftrightarrow \exists L \in \mho^n, y \in \operatorname{GR} \upuparrows f \cap \prod_{i \in n} \operatorname{atoms} \mathfrak{A}_i \forall i \in n: (y_i \sqsubseteq a_i \wedge y_i \sqsubseteq L_i) \Leftrightarrow \exists y \in \operatorname{GR} \upuparrows f \cap \prod_{i \in n} \operatorname{atoms} \mathfrak{A}_i \forall i \in n: y_i \sqsubseteq a_i \Leftrightarrow \operatorname{GR} \upuparrows f \cap \prod_{i \in n} \operatorname{atoms} a_i \neq \emptyset \Leftrightarrow a \in f$. □

Corollary 153. Let f is a completary staroid such that $(\operatorname{form} f)_i$ is a boolean lattice for each $i \in \operatorname{arity} f$. Let $a \in \prod_{i \in \operatorname{arity} f} \mathfrak{F}^{(\operatorname{form} f)_i}$. Then $\upuparrows f \not\asymp^{\mathsf{cStrd}(\operatorname{form} f)} \prod^{\mathsf{Strd}} a \Leftrightarrow a \in \upuparrows f$.