Really,

1. $L \in \operatorname{StarComp}(a; f) \Leftrightarrow \exists y \in a \forall i \in n: y_i \, f_i \, L_i$.

   Define the relation $R(f)$ by the formula $x R(f) \, y \Leftrightarrow \forall i \in n: x_i \, f_i \, y_i$. Obviously

   $$R(\lambda i \in n: g_i \circ f_i) = R(g) \circ R(f).$$

   $L \in \operatorname{StarComp}(a; f) \Leftrightarrow \exists y \in a: y R(f) \, L$.

   $L \in \operatorname{StarComp}(\operatorname{StarComp}(a; f); g) \Leftrightarrow \exists p \in \operatorname{StarComp}(a; f): p R(g) \; L \Leftrightarrow \exists p, \, y \in a$: $(y R(f) \; p \wedge p R(g) \; L) \Leftrightarrow \exists y \in a: y (R(g) \circ R(f)) \; L \Leftrightarrow \exists y \in a: (y R(\lambda i \in n: g_i \circ f_i) \; L) \Leftrightarrow$ $L \in \operatorname{StarComp}(a; \lambda i \in n: g_i \circ f_i)$ because $p \in \operatorname{StarComp}(a; f) \Leftrightarrow \exists y \in a: y R(f) \, p$.

2. Obvious.

3. It follows from the proposition above. □

Theorem 135. $\left\langle \prod^{(C)} f \right\rangle \prod a = \prod_{i \in n} \langle f_i \rangle a_i$ for every families $f = f_{i \in n}$ of binary relations and $a = a_{i \in n}$ where a_i is a small set *(for each $i \in n$).

Proof. $L \in \left\langle \prod^{(C)} f \right\rangle \prod a \Leftrightarrow L \in \operatorname{StarComp}(\prod a; f) \Leftrightarrow \exists y \in \prod a \forall i \in n: y_i \, f_i \, L_i \Leftrightarrow \exists y \in \prod a \forall i \in n$: $\{y\} \not\asymp \langle f_i^{-1} \rangle \{L_i\} \Leftrightarrow \forall i \in n \exists y \in a_i: \{y\} \not\asymp \langle f_i^{-1} \rangle \{L_i\} \Leftrightarrow \forall i \in n: a_i \not\asymp \langle f_i^{-1} \rangle \{L_i\} \Leftrightarrow \forall i \in n$: $\{L_i\} \not\asymp \langle f_i \rangle a_i \Leftrightarrow \forall i \in n: L_i \in \langle f_i \rangle a_i \Leftrightarrow L \in \prod_{i \in n} \langle f_i \rangle a_i$. □

11.5 Star composition of Rel-morphisms

Define *star composition* for an n-ary anchored relation a and an n-indexed family f of **Rel**-morphisms as an n-ary anchored relation complying with the formulas:

$$\operatorname{Obj}_{\operatorname{StarComp}(a; f)} = \lambda i \in \operatorname{arity} a: \operatorname{Dst} f_i;$$
$$\operatorname{arity} \operatorname{StarComp}(a; f) = \operatorname{arity} a;$$
$$L \in \operatorname{GR} \operatorname{StarComp}(a; f) \Leftrightarrow L \in \operatorname{StarComp}(\operatorname{GR} a; \operatorname{GR} \circ f).$$

(Here I denote $\operatorname{GR}(A; B; f) = f$ for every **Rel**-morphism f.)

Proposition 136. $b \not\asymp \operatorname{StarComp}(a; f) \Leftrightarrow \exists x \in a, y \in b \forall j \in n: x_j \, f_j \, y_j$.

Proof. From the previous section. □

Theorem 137. Relations with above defined compositions form a quasi-invertible category with star-morphisms.

Proof. We need to prove:

1. $\operatorname{StarComp}(\operatorname{StarComp}(m; f); g) = \operatorname{StarComp}(m; \lambda i \in \operatorname{arity} m: g_i \circ f_i)$;

2. $\operatorname{StarComp}(m; \lambda i \in \operatorname{arity} m: \operatorname{id}_{\operatorname{Obj}_m i}) = m$;

3. $b \not\asymp \operatorname{StarComp}(a; f) \Leftrightarrow a \not\asymp \operatorname{StarComp}(b; f^{\dagger})$

(the rest is obvious).

It follows from the previous section. □

Theorem 138. Cross-composition product of a family of **Rel**-morphisms is a discrete funcoid.

Proof. By the proposition and symmetry $\prod^{(C)} f$ is a pointfree funcoid. Obviously it is a funcoid $\prod_{i \in n} \operatorname{Src} f_i \rightarrow \prod_{i \in n} \operatorname{Dst} f_i$. Its completeness (and dually co-completeness) is obvious. □

11.6 Cross-composition product of funcoids

Let a is a an anchored relation of the form $\mathfrak{A}$ and $\operatorname{dom} \mathfrak{A} = n$.

Let every f_i (for all $i \in n$) is a pointfree funcoid with $\operatorname{Src} f_i = \mathfrak{A}_i$.