Remark 126. $\langle \chi f \rangle = (f \circ -)$ is the Hom-functor $\mathrm{Hom}(f, -)$ and we can apply Yoneda lemma to it.

Obvious 127. $\langle \chi(g \circ f) \rangle a = g \circ f \circ a$ for composable morphisms f and g or a quasi-invertible category.

11.2 General cross-composition

Let fix a quasi-invertible category with with star-morphisms. If f is an indexed family of morphisms from its base category, then the pointfree funcoid $\prod^{(C)} f$ from $\mathrm{StarHom}(\lambda i \in \mathrm{dom}\, f\colon \mathrm{Src}\, f_i)$ to $\mathrm{StarHom}(\lambda i \in \mathrm{dom}\, f\colon \mathrm{Dst}\, f_i)$ is defined by the formulas (for all star-morphisms a and b of these forms):

$$\left\langle \prod^{(C)} f \right\rangle a = \mathrm{StarComp}(a; f) \quad \text{and} \quad \left\langle \left(\prod^{(C)} f \right)^{-1} \right\rangle b = \mathrm{StarComp}(b; f^{\dagger}).$$

It is really a pointfree funcoid by the definition of quasi-invertible category.

In the terms of abrupt categories, these formulas can be rewritten as:

$$\prod^{(C)} f = \chi f.$$

Theorem 128. $\left(\prod^{(C)} g \right) \circ \left(\prod^{(C)} f \right) = \prod^{(C)}_{i \in n} (g_i \circ f_i)$ for every n-indexed families f and g of composable morphisms of a quasi-invertible category with star-morphisms.

Proof. $\left\langle \prod^{(C)}_{i \in n} (g_i \circ f_i) \right\rangle a = \mathrm{StarComp}(a; \lambda i \in n\colon g_i \circ f_i) = \mathrm{StarComp}(\mathrm{StarComp}(a; f); g)$ and $\left\langle \left(\prod^{(C)} g \right) \circ \left(\prod^{(C)} f \right) \right\rangle a = \left\langle \prod^{(C)} g \right\rangle \left\langle \prod^{(C)} f \right\rangle a = \mathrm{StarComp}(\mathrm{StarComp}(a; f); g)$. □

Corollary 129. $\left(\prod^{(C)} f_{k-1} \right) \circ \ldots \circ \left(\prod^{(C)} f_0 \right) = \prod^{(C)}_{i \in n} (f_i(k-1) \circ \ldots \circ f_i(k))$ for every n-indexed families $f_0, \ldots, f_{n-1}, g_0, \ldots, g_{n-1}$ composable morphisms of a quasi-invertible category with star-morphisms.

Proof. By math induction. □

11.3 Some properties of staroids

Lemma 130. Let $A_0, A_1 \in (\mathscr{P}\mho)^n$ are two families of sets and $\delta \in \mathscr{P}((\mathscr{P}\mho)^n)$. Then

$$\delta \cap \prod_{i \in n} (A_0 i \sqcup A_1 i) \neq \emptyset \Leftrightarrow \exists c \in \{0, 1\}^n\colon \delta \cap \prod_{i \in n} A_{c(i)} i \neq \emptyset.$$

Proof. $f \in \prod_{i \in n} (A_0 i \sqcup A_1 i) \Leftrightarrow \forall i \in n\colon (f_i \in A_0 i \cup A_1 i) \Leftrightarrow \forall i \in n\colon (f_i \in A_0 i \vee f_i \in A_1 i) \Leftrightarrow \exists c \in \{0, 1\}^n \forall i \in n\colon f_i \in A_{c(i)} i \Leftrightarrow \exists c \in \{0, 1\}^n\colon f \in \prod_{i \in n} A_{c(i)} i$.

$f \in \delta \cap \prod_{i \in n} (A_0\, i \sqcup A_1\, i) \Leftrightarrow f \in \delta \wedge \exists c \in \{0, 1\}^n\colon f \in \prod_{i \in n} A_{c(i)} i \Leftrightarrow \exists c \in \{0, 1\}^n\colon f \in \delta \cap \prod_{i \in n} A_{c(i)} i \Rightarrow \exists c \in \{0, 1\}^n\colon \delta \cap \prod_{i \in n} A_{c(i)}\, i \neq \emptyset$. The reverse implication is obvious. □

Theorem 131. Let $\mathfrak{A} = \mathfrak{A}_{i \in n}$ is a family of boolean lattices.

A relation $\delta \in \mathscr{P} \prod \mathrm{atoms}^{\mathfrak{F}(\mathfrak{A}_i)}$ such that for every $a \in \prod \mathrm{atoms}^{\mathfrak{F}(\mathfrak{A}_i)}$

$$\forall A \in a\colon \delta \cap \prod_{i \in n} \mathrm{atoms} \uparrow^{\mathfrak{A}_i} A_i \neq \emptyset \Rightarrow a \in \delta \tag{5}$$

can be continued till the function $\upuparrows f$ for a unique staroid f of the form $\lambda i \in n\colon \mathfrak{P}(\mathfrak{A}_i)$. The funcoid f is completary.

For every $\mathcal{X} \in \prod_{i \in n} \mathfrak{F}(\mathfrak{A}_i)$

$$\mathcal{X} \in \mathrm{GR} \upuparrows f \Leftrightarrow \delta \cap \prod_{i \in n} \mathrm{atoms}\, \mathcal{X}_i \neq \emptyset. \tag{6}$$