**Remark 122.** We can carry definitions (such as below defined cross-composition product) from categories with star-morphisms into plain dagger categories. This allows us to research properties of cross-composition product of indexed families of morphism for categories with star-morphisms without separately considering the special case of dagger categories and just binary star-composition product.

### 10.1 Abrupt of quasi-invertible categories with star-morphisms

**Definition 123.** The abrupt partially ordered pre-category of a partially ordered pre-category with star-morphisms is the abrupt pre-category with the following order of morphisms:

- Indexed (by arity $m$ for some $m \in M$) families of morphisms of $C$ are ordered as function spaces of posets.
- Star-morphisms (which are morphisms $\mathrm{None} \to \mathrm{Obj}_m$ for some $m \in M$) are ordered in the same order as in the pre-category with star-morphisms.
- Morphisms $\mathrm{None} \to \mathrm{None}$ which are only the identity morphism ordered by the unique order on this one-element set.

We need to prove it is a partially ordered pre-category.

**Proof.** It trivally follows from the definition of partially ordered pre-category with star-morphisms. □

**Theorem 124.** When a pre-category with star-morphisms is quasi-invertible, the corresponding abrupt category is also quasi-invertible.

**Proof.** We need to prove: $g \circ f \not\asymp h \Leftrightarrow g \not\asymp h \circ f^\dagger$ (or equivalently $f^\dagger \circ g \not\asymp h \Leftrightarrow g \not\asymp f \circ h$) for all kinds of morphisms.

Consider the cases:

**$g = \mathbf{id}_{\mathbf{None}}$.**
Subcases:

**$g = h = \mathbf{id}_{\mathbf{None}}$.** Trivial.

**$g \in M$.** $g \circ f \not\asymp h \Leftrightarrow g \not\asymp h \Leftrightarrow g \not\asymp h \circ f^\dagger$.

**$g \in M$.**
$f^\dagger \circ g \not\asymp h \Leftrightarrow \mathrm{StarComp}(g; f^\dagger) \not\asymp h \Leftrightarrow g \not\asymp \mathrm{StarComp}(h; f) \Leftrightarrow g \not\asymp f \circ h$.

**$g$ is a family of morphism of $C$.**
$f^\dagger \circ g \not\asymp h \Leftrightarrow \exists i \in \mathrm{dom}\, g: f_i^\dagger \circ g_i \not\asymp h_i \Leftrightarrow \exists i \in \mathrm{dom}\, g: g_i \not\asymp f_i \circ h_i \Leftrightarrow g \not\asymp f \circ h$. □

## 11 Product of an arbitrary number of funcoids

In this section it will be defined a product of an arbitrary (possibly infinite) family of funcoids.

### 11.1 Mapping a morphism into a pointfree funcoid

**Definition 125.** Let's define the pointfree funcoid $\chi f$ for every morphism $f$ or a quasi-invertible category:

$$\langle \chi f \rangle a = f \circ a \quad \text{and} \quad \langle (\chi f)^{-1} \rangle b = f^\dagger \circ b.$$

We need to prove it is really a pointfree funcoid.

**Proof.** $b \not\asymp \langle \chi f \rangle a \Leftrightarrow b \not\asymp f \circ a \Leftrightarrow a \not\asymp f^\dagger \circ b \Leftrightarrow a \not\asymp \langle (\chi f)^{-1} \rangle b$. □