9 Infinite product of elements and filters

Definition 90. Let A_i is a family of elements of a family $\mathfrak{A}_i$ of posets. The *staroidal product* $\prod^{\mathsf{Strd}(\mathfrak{A})} A_i$ is defined by the formula (for every $L \in \prod \mathfrak{A}$)

$$\operatorname{form} \prod^{\mathsf{Strd}(\mathfrak{A})} A = \mathfrak{A} \quad \text{and} \quad L \in \operatorname{GR} \prod^{\mathsf{Strd}(\mathfrak{A})} A \Leftrightarrow \forall i \in \operatorname{dom} \mathfrak{A}: A_i \not\asymp L_i.$$

Theorem 91. Staroidal product is a completary staroid (if our posets are distributive lattices).

Proof. We need to prove

$$\forall i \in \operatorname{dom} \mathfrak{A}: A_i \not\asymp (L_0 i \sqcup L_1 i) \Leftrightarrow \exists c \in \{0,1\}^n \forall i \in \operatorname{dom} \mathfrak{A}: A_i \not\asymp L_{c(i)} i.$$

Really, $\forall i \in \operatorname{dom} \mathfrak{A}: A_i \not\asymp (L_0 i \sqcup L_1 i) \Leftrightarrow \forall i \in \operatorname{dom} \mathfrak{A}: (A_i \not\asymp L_0 i \vee A_i \not\asymp L_1 i) \Leftrightarrow \exists c \in \{0,1\}^{\operatorname{dom} \mathfrak{A}} \forall i \in \operatorname{dom} \mathfrak{A}: A_i \not\asymp L_{c(i)} i$. □

Definition 92. Let $\mathfrak{A}$ is an indexed family of posets with least elements. Then *funcoidal product* is defined by the formulas:

$$\operatorname{form} \prod^{\mathsf{FCD}(\mathfrak{A})} A = \mathfrak{A} \quad \text{and} \quad \operatorname{GR}\left(\prod^{\mathsf{FCD}(\mathfrak{A})} A\right)_k L = \begin{cases} A_k & \text{if } \forall i \in (\operatorname{dom} \mathfrak{A}) \setminus \{k\}: A_i \not\asymp L_i \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 93. $\prod^{\mathsf{Strd}(\mathfrak{A})} A = \left[\prod^{\mathsf{FCD}(\mathfrak{A})} A\right]$.

Proof. $L \in \operatorname{GR} \prod^{\mathsf{Strd}(\mathfrak{A})} A \Leftrightarrow \forall i \in \operatorname{dom} \mathfrak{A}: A_i \not\asymp L_i \Leftrightarrow \forall i \in (\operatorname{dom} \mathfrak{A}) \setminus \{k\}: A_i \not\asymp L_i \wedge L_k \not\asymp A_k \Leftrightarrow A_k \not\asymp \left(\prod^{\mathsf{FCD}(\mathfrak{A})} A\right)_k L \Leftrightarrow L \in \operatorname{GR}\left[\prod^{\mathsf{FCD}(\mathfrak{A})} A\right]$. □

Corollary 94. Funcoidal product is a completary multifuncoid.

Proof. It is enough to prove that funcoidal product is a pre-multifuncoid. Really,

$$L_i \not\asymp \operatorname{GR}\left(\prod^{\mathsf{FCD}(\mathfrak{A})} A\right)_i L|_{(\operatorname{dom} \mathfrak{A}) \setminus \{i\}} \Leftrightarrow \forall i \in \operatorname{dom} \mathfrak{A}: A_i \not\asymp L_i \Leftrightarrow L_j \not\asymp \operatorname{GR}\left(\prod^{\mathsf{FCD}(\mathfrak{A})} A\right)_j L|_{(\operatorname{dom} \mathfrak{A}) \setminus \{j\}}. \quad \square$$

Theorem 95. If our filtrator $(\prod \mathfrak{A}; \prod \mathfrak{Z})$ is with separable core and $A \in \prod \mathfrak{Z}$, then $\upuparrows \prod^{\mathsf{Strd}(\mathfrak{Z})} A = \prod^{\mathsf{Strd}(\mathfrak{A})} A$.

Proof. $\operatorname{GR} \upuparrows \prod^{\mathsf{Strd}(\mathfrak{Z})} A = \left\{L \in \mathfrak{A} \mid L \subseteq \prod^{\mathsf{Strd}(\mathfrak{Z})} A\right\} = \{L \in \mathfrak{A} \mid \forall K \in L, i \in \operatorname{dom} \mathfrak{A}: A_i \not\asymp K_i\} = \{L \in \mathfrak{A} \mid \forall i \in \operatorname{dom} \mathfrak{A}, K \in L_i: A_i \not\asymp K\} = \{L \in \mathfrak{A} \mid \forall i \in \operatorname{dom} \mathfrak{A}: A_i \not\asymp L_i\} = \operatorname{GR} \prod^{\mathsf{Strd}(\mathfrak{A})} A$. □

Proposition 96. Let $(\prod \mathfrak{A}; \prod \mathfrak{Z})$ is a meet-closed filtrator. Then $\downdownarrows \prod^{\mathsf{Strd}(\mathfrak{A})} A = \prod^{\mathsf{Strd}(\mathfrak{Z})} A$.

Proof. $\operatorname{GR} \downdownarrows \prod^{\mathsf{Strd}(\mathfrak{A})} A = \downdownarrows \operatorname{GR} \prod^{\mathsf{Strd}(\mathfrak{A})} A = \downdownarrows \{L \in \prod \mathfrak{A} \mid \forall i \in \operatorname{dom} \mathfrak{A}: A_i \not\asymp L_i\} = \{L \in \prod \mathfrak{A} \mid \forall i \in \operatorname{dom} \mathfrak{A}: A_i \not\asymp L_i\} \cap \prod \mathfrak{Z} = \{L \in \prod \mathfrak{Z} \mid \forall i \in \operatorname{dom} \mathfrak{A}: A_i \not\asymp L_i\} = \operatorname{GR} \prod^{\mathsf{Strd}(\mathfrak{Z})} A$. □

Theorem 97. Let $\mathfrak{F}$ is a family of sets of filters on distributive lattices with least elements. Let $a \in \prod \mathfrak{F}$, $S \in \mathscr{P} \prod \mathfrak{F}$ is a generalized filter base, $\bigsqcap S = a$. Then

$$\prod^{\mathsf{Strd}(\mathfrak{F})} a = \bigsqcap \left\{ \prod^{\mathsf{Strd}(\mathfrak{F})} A \mid A \in S \right\}.$$

Proof. That $\prod^{\mathsf{Strd}(\mathfrak{F})} a$ is a lower bound for $\left\{\prod^{\mathsf{Strd}(\mathfrak{F})} A \mid A \in S\right\}$ is obvious.