Remark 82. To describe this, the definition of order poset is used twice. Let f and g are posets of the same form $\mathfrak{A}$

$$\langle f\rangle \sqsubseteq \langle g\rangle \Leftrightarrow \forall i\in \operatorname{dom}\mathfrak{A}: \langle f\rangle_i \sqsubseteq \langle g\rangle_i \quad \text{and} \quad \langle f\rangle_i \sqsubseteq \langle g\rangle_i \Leftrightarrow \forall L\in \prod \mathfrak{A}|_{(\operatorname{dom}\mathfrak{A})\setminus\{i\}}: \langle f\rangle_i L \sqsubseteq \langle g\rangle_i L.$$

Theorem 83. $f \sqcup^{\mathsf{pFCD}(\mathfrak{A})} g = f \sqcup g$ for every pre-multifuncoids f and g of the same form $\mathfrak{A}$ of distributive lattices.

Proof. $\alpha_i x \overset{\text{def}}{=} f_i x \sqcup g_i x$. It is enough to prove that α is a multifuncoid.

We need to prove:

$$L_i \not\asymp \alpha_i L|_{(\operatorname{dom} L)\setminus\{i\}} \Leftrightarrow L_j \not\asymp \alpha_j L|_{(\operatorname{dom} L)\setminus\{j\}}.$$

Really, $L_i \not\asymp \alpha_i\ L|_{(\operatorname{dom} L)\setminus\{i\}} \Leftrightarrow L_i \not\asymp f_i\ L|_{(\operatorname{dom} L)\setminus\{i\}} \sqcup g_i\ L|_{(\operatorname{dom} L)\setminus\{i\}} \Leftrightarrow L_i \not\asymp f_i\ L|_{(\operatorname{dom} L)\setminus\{i\}} \vee L_i \not\asymp g_i\ L|_{(\operatorname{dom} L)\setminus\{i\}} \Leftrightarrow L_j \not\asymp f_j\ L|_{(\operatorname{dom} L)\setminus\{j\}} \vee L_j \not\asymp g_j\ L|_{(\operatorname{dom} L)\setminus\{j\}} \Leftrightarrow L_j \not\asymp f_j\ L|_{(\operatorname{dom} L)\setminus\{j\}} \sqcup g_j L|_{(\operatorname{dom} L)\setminus\{j\}} \Leftrightarrow L_j \not\asymp \alpha_j L|_{(\operatorname{dom} L)\setminus\{j\}}$. □

Theorem 84. $\bigsqcup^{\mathsf{pFCD}(\mathfrak{A})} F = \bigsqcup F$ for every set F of pre-multifuncoids of the same form $\mathfrak{A}$ of join infinite distributive complete lattices.

Proof. $\alpha_i x \overset{\text{def}}{=} \bigsqcup_{f\in F} f_i x$. It is enough to prove that α is a multifuncoid.

We need to prove:

$$L_i \not\asymp \alpha_i L|_{(\operatorname{dom} L)\setminus\{i\}} \Leftrightarrow L_j \not\asymp \alpha_j L|_{(\operatorname{dom} L)\setminus\{j\}}.$$

Really, $L_i \not\asymp \alpha_i L|_{(\operatorname{dom} L)\setminus\{i\}} \Leftrightarrow L_i \not\asymp \bigsqcup_{f\in F} f_i L|_{(\operatorname{dom} L)\setminus\{i\}} \Leftrightarrow \exists f\in F: L_i \not\asymp f_i L|_{(\operatorname{dom} L)\setminus\{i\}} \Leftrightarrow \exists f\in F: L_j \not\asymp f_j L|_{(\operatorname{dom} L)\setminus\{j\}} \Leftrightarrow L_j \not\asymp \bigsqcup_{f\in F} f_j L|_{(\operatorname{dom} L)\setminus\{j\}} \Leftrightarrow L_j \not\asymp \alpha_j L|_{(\operatorname{dom} L)\setminus\{j\}}$. □

Proposition 85. The mapping $f\mapsto [f]$ is an order embedding, for multifuncoids of the form $\mathfrak{A}$ of separable starrish posets.

Proof. The mapping $f\mapsto [f]$ is defined because $\mathfrak{A}$ are starrish poset. The mapping is injective because $\mathfrak{A}$ are separable posets. That $f\mapsto [f]$ is a monotone function is obvious. □

Remark 86. This order embedding is useful to describe properties of posets of pre-staroids.

Theorem 87. If f, g are multifuncoids of the same form $\mathfrak{A}$ of distributive lattices, then $f \sqcup^{\mathsf{pFCD}(\mathfrak{A})} g \in \mathsf{FCD}(\mathfrak{A})$.

Proof. Let $A\in \left[f \sqcup^{\mathsf{pFCD}(\mathfrak{A})} g\right]$ and $B \sqsupseteq A$. Then for every $k\in\operatorname{dom}\mathfrak{A}$

$A_k \not\asymp \left(f \sqcup^{\mathsf{pFCD}(\mathfrak{A})} g\right) A|_{(\operatorname{dom}\mathfrak{A})\setminus\{k\}} = (f\sqcup g) A|_{(\operatorname{dom}\mathfrak{A})\setminus\{k\}} = f(A|_{(\operatorname{dom}\mathfrak{A})\setminus\{k\}}) \sqcup g(A|_{(\operatorname{dom}\mathfrak{A})\setminus\{k\}})$.

Thus $A_k \not\asymp f(A|_{(\operatorname{dom}\mathfrak{A})\setminus\{k\}}) \vee A_k \not\asymp g(A|_{(\operatorname{dom}\mathfrak{A})\setminus\{k\}})$; $A\in [f] \vee A\in [g]$; $B\in [f]\vee B\in [g]$; $B_k \not\asymp f(B|_{(\operatorname{dom}\mathfrak{A})\setminus\{k\}}) \vee B_k \not\asymp g(B|_{(\operatorname{dom}\mathfrak{A})\setminus\{k\}})$; $f(B|_{(\operatorname{dom}\mathfrak{A})\setminus\{k\}}) \sqcup g(B|_{(\operatorname{dom}\mathfrak{A})\setminus\{k\}}) = (f\sqcup g) B|_{(\operatorname{dom}\mathfrak{A})\setminus\{k\}} = \left(f \sqcup^{\mathsf{pFCD}(\mathfrak{A})} g\right) B|_{(\operatorname{dom}\mathfrak{A})\setminus\{k\}} \not\asymp B_k$. Thus $B\in \left[f \sqcup^{\mathsf{pFCD}(\mathfrak{A})} g\right]$. □

Theorem 88. If F is a set multifuncoids of the same form $\mathfrak{A}$ of join inifinite distributive complete lattices, then $\bigsqcup^{\mathsf{pFCD}(\mathfrak{A})} f \in \mathsf{FCD}(\mathfrak{A})$.

Proof. Let $A\in \left[\bigsqcup^{\mathsf{pFCD}(\mathfrak{A})} f\right]$ and $B\sqsupseteq A$. Then for every $k\in\operatorname{dom}\mathfrak{A}$.

$A_k \not\asymp \left(\bigsqcup^{\mathsf{pFCD}(\mathfrak{A})} F\right) A|_{(\operatorname{dom}\mathfrak{A})\setminus\{k\}} = \left(\bigsqcup F\right) A|_{(\operatorname{dom}\mathfrak{A})\setminus\{k\}} = \bigsqcup_{f\in F} f(A|_{(\operatorname{dom}\mathfrak{A})\setminus\{k\}})$.

Thus $\exists f\in F: A_k \not\asymp f(A|_{(\operatorname{dom}\mathfrak{A})\setminus\{k\}})$; $\exists f\in F: A\in [f]$; $B\in [f]\vee B\in [g]$; $\exists f\in F: B_k \not\asymp f(B|_{(\operatorname{dom}\mathfrak{A})\setminus\{k\}})$; $\bigsqcup_{f\in F} f(B|_{(\operatorname{dom}\mathfrak{A})\setminus\{k\}}) = (f\sqcup g) B|_{(\operatorname{dom}\mathfrak{A})\setminus\{k\}} = \left(\bigsqcup^{\mathsf{pFCD}(\mathfrak{A})} F\right) B|_{(\operatorname{dom}\mathfrak{A})\setminus\{k\}} \not\asymp B_k$. Thus $B\in \left[\bigsqcup^{\mathsf{pFCD}(\mathfrak{A})} F\right]$. □

Conjecture 89. The formula $f \sqcup^{\mathsf{FCD}(\mathfrak{A})} g \in \mathsf{cFCD}(\mathfrak{A})$ is not true in general for completary multifuncoids (even for multifuncoids on powersets) f and g of the same form $\mathfrak{A}$.