3.2 Definition

Definition 4. I will call a *quasi-invertible category* a partially ordered dagger category such that it holds

$$g \circ f \not\asymp h \Leftrightarrow g \not\asymp h \circ f^{\dagger} \tag{1}$$

for every morphisms $f \in \mathrm{Hom}(A; B)$, $g \in \mathrm{Hom}(B; C)$, $h \in \mathrm{Hom}(A; C)$, where A, B, C are objects of this category.

Inverting this formula, we get $f^{\dagger} \circ g^{\dagger} \not\asymp h^{\dagger} \Leftrightarrow g^{\dagger} \not\asymp f \circ h^{\dagger}$. After replacement of variables, this gives: $f^{\dagger} \circ g \not\asymp h \Leftrightarrow g \not\asymp f \circ h$

As it follows from [1], the category of funcoids and the category of reloids are quasi-invertible (taking $f^{\dagger} = f^{-1}$). Moreover by [3] the category of pointfree funcoids between lattices of filters on boolean lattices are quasi-invertible.

Definition 5. The *cross-composition product* of morphisms f and g of a quasi-invertible category is the pointfree funcoid $\mathrm{Hom}(\mathrm{Src}\, f; \mathrm{Src}\, g) \rightarrow \mathrm{Hom}(\mathrm{Dst}\, f; \mathrm{Dst}\, g)$ defined by the formulas (for every $a \in \mathrm{Hom}(\mathrm{Src}\, f; \mathrm{Src}\, g)$ and $b \in \mathrm{Hom}(\mathrm{Dst}\, f; \mathrm{Dst}\, g)$):

$$\left\langle f \times^{(C)} g \right\rangle a = g \circ a \circ f^{\dagger} \quad \text{and} \quad \left\langle \left(f \times^{(C)} g\right)^{-1} \right\rangle b = g^{\dagger} \circ b \circ f.$$

The cross-composition product is a pointfree funcoid from $\mathrm{Hom}(\mathrm{Src}\, f; \mathrm{Src}\, g)$ to $\mathrm{Hom}(\mathrm{Dst}\, f; \mathrm{Dst}\, g)$.

We need to prove that it is really a pointfree funcoid that is that

$$b \not\asymp \left\langle f \times^{(C)} g \right\rangle a \Leftrightarrow a \not\asymp \left\langle \left(f \times^{(C)} g\right)^{-1} \right\rangle b.$$

This formula means $b \not\asymp g \circ a \circ f^{\dagger} \Leftrightarrow a \not\asymp g^{\dagger} \circ b \circ f$ and can be easily proved applying the formula (1) two times.

Proposition 6. $a \left[f \times^{(C)} g \right] b \Leftrightarrow a \circ f^{\dagger} \not\asymp g^{\dagger} \circ b$.

Proof. From the lemma. □

Proposition 7. $a \left[f \times^{(C)} g \right] b \Leftrightarrow f \left[a \times^{(C)} b \right] g$.

Proof. $f \left[a \times^{(C)} b \right] g \Leftrightarrow f \circ a^{\dagger} \not\asymp b^{\dagger} \circ g \Leftrightarrow a \circ f^{\dagger} \not\asymp g^{\dagger} \circ b \Leftrightarrow a \left[f \times^{(C)} g \right] b$. □

Theorem 8. $\left(f \times^{(C)} g\right)^{\dagger} = f^{\dagger} \times^{(C)} g^{\dagger}$.

Proof. For every funcoids $a \in \mathrm{Hom}(\mathrm{Src}\, f; \mathrm{Src}\, g)$ and $b \in \mathrm{Hom}(\mathrm{Dst}\, f; \mathrm{Dst}\, g)$ we have:

$\left\langle \left(f \times^{(C)} g\right)^{\dagger} \right\rangle b = g^{\dagger} \circ b \circ f = g^{\dagger} \circ b \circ f = \left\langle f^{\dagger} \times^{(C)} g^{\dagger} \right\rangle b$.

$\left\langle \left(\left(f \times^{(C)} g\right)^{\dagger}\right)^{\dagger} \right\rangle a = \left\langle f \times^{(C)} g \right\rangle a = g \circ a \circ f^{\dagger} = \left\langle \left(f^{\dagger} \times^{(C)} g^{\dagger}\right)^{\dagger} \right\rangle a$. □

Theorem 9. Let f, g are morphisms of a quasi-invertible category where $\mathrm{Dst}\, f$ and $\mathrm{Dst}\, g$ are f.o. on boolean lattices. Then for every f.o. $\mathcal{A}_0 \in \mathfrak{F}(\mathrm{Src}\, f)$, $\mathcal{B}_0 \in \mathfrak{F}(\mathrm{Src}\, g)$

$$\left\langle f \times^{(C)} g \right\rangle (\mathcal{A}_0 \times^{\mathsf{FCD}} \mathcal{B}_0) = \langle f \rangle \mathcal{A}_0 \times^{\mathsf{FCD}} \langle g \rangle \mathcal{B}_0.$$

Proof. For every atom $a_1 \times^{\mathsf{FCD}} b_1$ ($a_1 \in \mathrm{atoms}^{\mathrm{Dst}\, f}$, $b_1 \in \mathrm{atoms}^{\mathrm{Dst}\, g}$) of the lattice of funcoids we have:

$a_1 \times^{\mathsf{FCD}} b_1 \not\asymp \left\langle f \times^{(C)} g \right\rangle (\mathcal{A}_0 \times^{\mathsf{FCD}} \mathcal{B}_0) \Leftrightarrow \mathcal{A}_0 \times^{\mathsf{FCD}} \mathcal{B}_0 \left[f \times^{(C)} g \right] a_1 \times^{\mathsf{FCD}} b_1 \Leftrightarrow (\mathcal{A}_0 \times^{\mathsf{FCD}} \mathcal{B}_0) \circ f^{\dagger} \not\asymp g^{\dagger} \circ (a_1 \times^{\mathsf{FCD}} b_1) \Leftrightarrow \langle f \rangle \mathcal{A}_0 \times^{\mathsf{FCD}} \mathcal{B}_0 \not\asymp a_1 \times^{\mathsf{FCD}} \langle g^{\dagger} \rangle b_1 \Leftrightarrow \langle f \rangle \mathcal{A}_0 \not\asymp a_1 \wedge \langle g^{\dagger} \rangle b_1 \not\asymp \mathcal{B}_0 \Leftrightarrow \langle f \rangle \mathcal{A}_0 \not\asymp a_1 \wedge \langle g \rangle \mathcal{B}_0 \not\asymp b_1 \Leftrightarrow \langle f \rangle \mathcal{A}_0 \times^{\mathsf{FCD}} \langle g \rangle \mathcal{B}_0 \not\asymp a_1 \times^{\mathsf{FCD}} b_1$. Thus $\left\langle f \times^{(C)} g \right\rangle (\mathcal{A}_0 \times^{\mathsf{FCD}} \mathcal{B}_0) = \langle f \rangle \mathcal{A}_0 \times^{\mathsf{FCD}} \langle g \rangle \mathcal{B}_0$ because the lattice $\mathsf{FCD}(\mathfrak{F}(\mathrm{Dst}\, f); \mathfrak{F}(\mathrm{Dst}\, g))$ is atomically separable (corollary 64 in [3]). □

Proposition 10. $\mathcal{A}_0 \times^{\mathsf{FCD}} \mathcal{B}_0 \left[f \times^{(C)} g \right] \mathcal{A}_1 \times^{\mathsf{FCD}} \mathcal{B}_1 \Leftrightarrow \mathcal{A}_0 [f] \mathcal{A}_1 \wedge \mathcal{B}_0 [g] \mathcal{B}_1$ for every $\mathcal{A}_0 \in \mathfrak{F}(\mathrm{Src}\, f)$, $\mathcal{A}_1 \in \mathfrak{F}(\mathrm{Dst}\, f)$, $\mathcal{B}_0 \in \mathfrak{F}(\mathrm{Src}\, g)$, $\mathcal{B}_1 \in \mathfrak{F}(\mathrm{Dst}\, g)$.

Proof. $\mathcal{A}_0 \times^{\mathsf{FCD}} \mathcal{B}_0 \left[f \times^{(C)} g \right] \mathcal{A}_1 \times^{\mathsf{FCD}} \mathcal{B}_1 \Leftrightarrow \mathcal{A}_1 \times^{\mathsf{FCD}} \mathcal{B}_1 \not\asymp \left\langle f \times^{(C)} g \right\rangle (\mathcal{A}_0 \times^{\mathsf{FCD}} \mathcal{B}_0) \Leftrightarrow \mathcal{A}_1 \times^{\mathsf{FCD}} \mathcal{B}_1 \not\asymp \langle f \rangle \mathcal{A}_0 \times^{\mathsf{FCD}} \langle g \rangle \mathcal{B}_0 \Leftrightarrow \mathcal{A}_1 \not\asymp \langle f \rangle \mathcal{A}_0 \wedge \mathcal{B}_1 \not\asymp \langle g \rangle \mathcal{B}_0 \Leftrightarrow \mathcal{A}_0 [f] \mathcal{A}_1 \wedge \mathcal{B}_0 [g] \mathcal{B}_1$. □