We are going to consider (generalized) limits of arbitrary functions acting from μ to ν. (The functions in consideration are not required to be continuous.)

Remark 12. Most typically G is the group of translations of some topological vector space.

Generalized limit is defined by the following formula:

Definition 13. $\operatorname{xlim} f \overset{\text{def}}{=} \{\nu \circ f \circ r \mid r \in G\}$ *for any funcoid* f.

Remark 14. Generalized limit technically is a set of funcoids (see [2]).

We will assume that the function f is defined on $\langle\mu\rangle\{x\}$.

Definition 15. $\operatorname{xlim}_x f \overset{\text{def}}{=} \operatorname{xlim} f|_{\langle\mu\rangle\{x\}}$.

Obvious 16. $\operatorname{xlim}_x f = \{\nu \circ f|_{\langle\mu\rangle\{x\}} \circ r \mid r \in G\}$.

Remark 17. $\operatorname{xlim}_x f$ is the same for funcoids μ and $\operatorname{Compl}\mu$.

Lemma 18. $(\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) \circ f = \langle f^{-1}\rangle\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}$ *for every* $f \in \mathsf{FCD}$, $\mathcal{A}, \mathcal{B} \in \mathfrak{F}$.

Proof. For every filter object $\mathcal{X}$

$$\begin{aligned} \langle(\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) \circ f\rangle\mathcal{X} &= \\ \langle(\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B})\rangle\langle f\rangle\mathcal{X} &= \\ \begin{cases} \mathcal{B} & \text{if } \langle f\rangle\mathcal{X} \cap \mathcal{A} \neq \emptyset; \\ \emptyset & \text{if } \langle f\rangle\mathcal{X} \cap \mathcal{A} = \emptyset; \end{cases} &= \\ \begin{cases} \mathcal{B} & \text{if } \mathcal{X} \cap \langle f^{-1}\rangle\mathcal{A} \neq \emptyset; \\ \emptyset & \text{if } \mathcal{X} \cap \langle f^{-1}\rangle\mathcal{A} = \emptyset; \end{cases} &= \\ \langle\langle f^{-1}\rangle\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}\rangle\mathcal{X}. & \end{aligned}$$

□

The function τ will define an injection from the set of points of the space ν ("numbers", "points", or "vectors") to the set of all (generalized) limits (i.e. values which $\operatorname{xlim}_x f$ may take).

Definition 19. $\tau(y) \overset{\text{def}}{=} \{\langle\mu\rangle\{x\} \times^{\mathsf{FCD}} \langle\nu\rangle\{y\} \mid x \in D\}$.

Proposition 20. $\tau(y) \overset{\text{def}}{=} \{(\langle\mu\rangle\{x\} \times^{\mathsf{FCD}} \langle\nu\rangle\{y\}) \circ r \mid r \in G\}$ *for every* $x \in D$.

Proof. $(\langle\mu\rangle\{x\} \times^{\mathsf{FCD}} \langle\nu\rangle y) \circ r = \langle r^{-1}\rangle\langle\mu\rangle\{x\} \times^{\mathsf{FCD}} \langle\nu\rangle y = \langle\mu\rangle\langle r^{-1}\rangle\{x\} \times^{\mathsf{FCD}} \langle\nu\rangle y = \langle\mu\rangle\{r^{-1}x\} \times^{\mathsf{FCD}} \langle\nu\rangle y \in \{\langle\mu\rangle\{x\} \times^{\mathsf{FCD}} \langle\nu\rangle\{y\} \mid x \in D\}$ where $x' \in D$.

Reversely $\langle\mu\rangle\{x\} \times^{\mathsf{FCD}} \langle\nu\rangle\{y\} = (\langle\mu\rangle\{x\} \times^{\mathsf{FCD}} \langle\nu\rangle\{y\}) \circ e$ where e is the identify element of G. □

Theorem 21. *If* $f|_{\langle\mu\rangle\{x\}} \in \mathrm{C}(\mu;\nu)$ *and* $\langle\mu\rangle\{x\} \supseteq \{x\}$ *then* $\operatorname{xlim}_x f = \tau(fx)$.

Proof. $f|_{\langle\mu\rangle\{x\}} \circ \mu \subseteq \nu \circ f|_{\langle\mu\rangle\{x\}} \subseteq \nu \circ f$; thus $\langle f\rangle\langle\mu\rangle\{x\} \subseteq \langle\nu\rangle\langle f\rangle\{x\}$; consequently we have

$$\nu \supseteq \langle\nu\rangle\langle f\rangle\{x\} \times^{\mathsf{FCD}} \langle\nu\rangle\langle f\rangle\{x\} \supseteq \langle f\rangle\langle\mu\rangle\{x\} \times^{\mathsf{FCD}} \langle\nu\rangle\langle f\rangle\{x\}.$$

$$\begin{aligned} \nu \circ f|_{\langle\mu\rangle\{x\}} &\supseteq \\ (\langle f\rangle\langle\mu\rangle\{x\} \times^{\mathsf{FCD}} \langle\nu\rangle\langle f\rangle\{x\}) \circ f|_{\langle\mu\rangle\{x\}} &= \\ \langle(f|_{\langle\mu\rangle\{x\}})^{-1}\rangle\langle f\rangle\langle\mu\rangle\{x\} \times^{\mathsf{FCD}} \langle\nu\rangle\langle f\rangle\{x\} &\supseteq \\ \operatorname{dom} f|_{\langle\mu\rangle\{x\}} \times^{\mathsf{FCD}} \langle\nu\rangle\langle f\rangle\{x\} &= \\ \langle\mu\rangle\{x\} \times^{\mathsf{FCD}} \langle\nu\rangle\langle f\rangle\{x\}. & \end{aligned}$$