Algebraic General Topology (theory of funcoids and reloids)

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In my book [1] I introduce some new concepts generalizing general topology, including *funcoids* and *reloids*. The book along with supplementary materials (such as a partial draft of the second volume) is freely available (including the LaTeX source) online.

Before studying funcoids, in my book I consider co-brouwerian lattices and lattices of filters in particular, as the theory of funcoids is based on theory of filters. My book contains probably the best (and most detailed) published overview of properties of filters.

Because you always can refer to my book, in this short intro I present theorems without proofs.

I denote order on a poset as $\sqsubseteq$ and corresponding lattice operations as $\bigsqcup$ and $\bigsqcap$. I denote the least and greatest elements (if they exist) of our poset as $\bot$ and $\top$ correspondingly.

1. Filters

Definition 2. I order the set $\mathfrak{F}$ of filters (including the improper filter) *reverse* to set-theoretic order, that is $\mathcal{A} \sqsubseteq \mathcal{B} \Leftrightarrow \mathcal{A} \supseteq \mathcal{B}$ for $\mathcal{A}, \mathcal{B} \in \mathfrak{F}$.

Proposition 3. *This makes the set of filters on a set into a co-brouwerian (and thus distributive) lattice, that is we have $\mathcal{A} \sqcup \bigsqcap S = \bigsqcap_{\mathcal{X} \in S}(\mathcal{A} \sqcup \mathcal{X})$ for a set S of filters and a filter $\mathcal{A}$.*

4. Funcoids

Let $\mathfrak{F}(A)$, $\mathfrak{F}(B)$ be sets of filters on sets A, B. They are complete atomistic co-brouwerian lattices.

Definition 5. A *funcoid* $A \to B$ is a quadruple (A, B, α, β) where α and β are functions $\mathfrak{F}(A) \to \mathfrak{F}(B)$ and $\mathfrak{F}(B) \to \mathfrak{F}(A)$ correspondingly, such that $\mathcal{Y} \sqcap \alpha(\mathcal{X}) \neq \bot \Leftrightarrow \mathcal{X} \sqcap \beta(\mathcal{Y}) \neq \bot$ for every $\mathcal{X} \in \mathfrak{F}(A)$, $\mathcal{Y} \in \mathfrak{F}(B)$.

Definition 6. I denote $(A, B, \alpha, \beta)^{-1} = (B, A, \beta, \alpha)$.

Definition 7. I denote $\langle (A, B, \alpha, \beta) \rangle = \alpha$.

Funcoids generalize such things as:

- binary relations;
- proximity spaces;
- pretopologies;
- preclosures.

For a proximity δ, define

$$\mathcal{X}\ \delta'\ \mathcal{Y} \Leftrightarrow \forall X \in \mathcal{X}, Y \in \mathcal{Y} : X\ \delta\ Y$$

for all filters $\mathcal{X}$, $\mathcal{Y}$. Then we have a unique funcoid f such that

$$\mathcal{X}\ \delta'\ \mathcal{Y} \Leftrightarrow \mathcal{Y} \sqcap \langle f \rangle \mathcal{X} \neq \bot \Leftrightarrow \mathcal{X} \sqcap \langle f^{-1} \rangle \mathcal{Y} \neq \bot.$$

Definition 8. $\mathcal{X}\ [f]\ \mathcal{Y} \Leftrightarrow \mathcal{Y} \sqcap \langle f \rangle \mathcal{X} \neq \bot \Leftrightarrow \mathcal{X} \sqcap \langle f^{-1} \rangle \mathcal{Y} \neq \bot$.

Proposition 9. *A funcoid $f : A \to B$ is uniquely determined by $\langle f \rangle$ and moreover is uniquely determined by values of the function $\langle f \rangle$ on principal filters or by the relation $[f]$ between principal filters.*