That state is not good.

3 The rationale and examples

In mathematics it is often encountered that a small set S naturally bijectively corresponds to a subset R of a larger set B. (In other words, there is specified an injection E from S to B.) It is a widespread practice to equate S with R.

Remark 1. I denote the first set S from the first letter of the word "small" and the second set B from the first letter of the word "big", because S is intuitively considered as smaller than B. (However we do not require $\operatorname{card} S < \operatorname{card} B$.) I denote the injection as E from the first letter of the word "embed" because it embeds the set S to the set B.

The set B is considered as a generalization of the set S, for example: whole numbers generalizing natural numbers, rational numbers generalizing whole numbers, real numbers generalizing rational numbers, complex numbers generalizing real numbers, etc.

Through these examples we see that B can be considered a generalization of S.

But strictly speaking this equating may contradict to the axioms of ZF/ZFC because we are not insured against $S \cap B \neq \emptyset$ incidents. Not wonderful, as it is often labeled as "without proof".

To work around of this (and formulate things exactly what could benefit computer proof assistants) we will replace the set B with a new set B' having a bijection $M: B \to B'$ such that $M \circ E = \mathrm{id}_S$. (I call this bijection M from the first letter of the word "move" which signifies the move from the old set B to a new set B').

4 Generalization situation

Now to the formalistic: I will call a *generalization situation* sets S and B together with an injection E from S to B. I will call, given a generalization situation, an *arbitrary generalization* a set B' and a bijection $M: B \to B'$ such that $M \circ E = \mathrm{id}_S$.

For every generalization situation I will denote $R = \operatorname{im} E$.

Proposition 2. *For every arbitrary generalization:*

1. $S \subseteq B'$.
2. *E is a bijection from S to R.*
3. $E = M^{-1}|_S$.

Proof.

1. $S = \operatorname{im}(\mathrm{id}_S) = \operatorname{im}(M \circ E) \subseteq \operatorname{im} M = B'$.
2. Obvious.
3. $M \circ E = \mathrm{id}_S \Rightarrow M^{-1} \circ M \circ E = M^{-1} \circ \mathrm{id}_S \Rightarrow E = M^{-1}|_S$. □

Assuming axiom of foundation (one of the axioms of ZF, also known as *axiom of regularity*) I will prove that an arbitrary generalization always exist for every generalization situation. Specifically I will prove that *ZF generalization* (see below) is an arbitrary generalization.

In absence of the axiom of foundation one could reasonably assert existence of a bijection $M: B \to B'$ such that $M \circ E = \mathrm{id}_S$ as an axiom. (Let's call it *the axiom of generalization*.) This axiom does not contradict to ZF because it is a consequence of the axiom of foundation.

5 ZF generalization

Let S and B are sets. Let E is an injection from S to B. (So we have a generalization situation.) Let $R = \operatorname{im} E$.