6.2 Our three definitions of continuity

I have expressed different kinds of continuity with simple algebraic formulas hiding the complexity of traditional epsilon-delta notation behind a smart algebra. Let's summarize these three algebraic formulas:

Let μ and ν are endomorphisms of some partially ordered precategory. Continuous functions can be defined as these morphisms f of this precategory which conform to the following formula:

$$f \in \mathrm{C}(\mu; \nu) \Leftrightarrow f \in \mathrm{Mor}(\mathrm{Ob}\,\mu; \mathrm{Ob}\,\nu) \wedge f \circ \mu \subseteq \nu \circ f.$$

If the precategory is a partially ordered dagger precategory then continuity also can be defined in two other ways:

$$\begin{aligned} f \in \mathrm{C}'(\mu; \nu) &\Leftrightarrow& f \in \mathrm{Mor}(\mathrm{Ob}\,\mu; \mathrm{Ob}\,\nu) \wedge \mu \subseteq f^\dagger \circ \nu \circ f; \\ f \in \mathrm{C}''(\mu; \nu) &\Leftrightarrow& f \in \mathrm{Mor}(\mathrm{Ob}\,\mu; \mathrm{Ob}\,\nu) \wedge f \circ \mu \circ f^\dagger \subseteq \nu. \end{aligned}$$

Remark 8 In the examples (above) about funcoids and reloids the "dagger functor" is the inverse of a funcoid or reloid, that is $f^\dagger = f^{-1}$.

Proposition 40 *Every of these three definitions of continuity forms a subprecategory (subcategory if the original precategory is a category).*

Proof

C Let $f \in \mathrm{C}(\mu; \nu)$, $g \in \mathrm{C}(\nu; \pi)$. Then $f \circ \mu \subseteq \nu \circ f$, $g \circ \nu \subseteq \pi \circ g$; $g \circ f \circ \mu \subseteq g \circ \nu \circ f \subseteq \pi \circ g \circ f$. So $g \circ f \in \mathrm{C}(\mu; \pi)$. $1_{\mathrm{Ob}\,\mu} \in \mathrm{C}(\mu; \mu)$ is obvious.

C' Let $f \in \mathrm{C}'(\mu; \nu)$, $g \in \mathrm{C}'(\nu; \pi)$. Then $\mu \subseteq f^\dagger \circ \nu \circ f$, $\nu \subseteq g^\dagger \circ \pi \circ g$;

$$\mu \subseteq f^\dagger \circ g^\dagger \circ \pi \circ g \circ f; \quad \mu \subseteq (g \circ f)^\dagger \circ \pi \circ (g \circ f).$$

So $g \circ f \in \mathrm{C}'(\mu; \pi)$. $1_{\mathrm{Ob}\,\mu} \in \mathrm{C}'(\mu; \mu)$ is obvious.

C″ Let $f \in \mathrm{C}''(\mu; \nu)$, $g \in \mathrm{C}''(\nu; \pi)$. Then $f \circ \mu \circ f^\dagger \subseteq \nu$, $g \circ \nu \circ g^\dagger \subseteq \pi$;

$$g \circ f \circ \mu \circ f^\dagger \circ g^\dagger \subseteq \pi; \quad (g \circ f) \circ \mu \circ (g \circ f)^\dagger \subseteq \pi.$$

So $g \circ f \in \mathrm{C}''(\mu; \pi)$. $1_{\mathrm{Ob}\,\mu} \in \mathrm{C}''(\mu; \mu)$ is obvious.

□

Proposition 41 *For a monovalued morphism f of a partially ordered dagger category and its endomorphisms μ and ν*

$$f \in \mathrm{C}'(\mu; \nu) \Rightarrow f \in \mathrm{C}(\mu; \nu) \Rightarrow f \in \mathrm{C}''(\mu; \nu).$$