5.2 Reloids induced by funcoid

Every funcoid $f \in \mathsf{FCD}(A;B)$ induces a reloid from A to B in two ways, intersection of *outward* relations and union of *inward* direct products of filter objects:

$$(\mathsf{RLD})_{\mathrm{out}} f \overset{\mathrm{def}}{=} \bigcap \left\langle \uparrow^{\mathsf{RLD}(A;B)} \right\rangle \mathrm{up}\, f;$$
$$(\mathsf{RLD})_{\mathrm{in}} f \overset{\mathrm{def}}{=} \bigcup \left\{ \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B} \quad | \quad \mathcal{A} \in \mathfrak{F}(A), \mathcal{B} \in \mathfrak{F}(B), \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \subseteq f \right\}.$$

Theorem 64 $(\mathsf{RLD})_{\mathrm{in}} f = \bigcup \left\{ a \times^{\mathsf{RLD}} b \quad | \quad a \in \mathrm{atoms}\, 1^{\mathfrak{F}(\mathrm{Src}\, f)}, b \in \mathrm{atoms}\, 1^{\mathfrak{F}(\mathrm{Dst}\, f)}, a \times^{\mathsf{FCD}} b \subseteq f \right\}$.

Proof It follows from the theorem 46. □

Remark 6 It seems that $(\mathsf{RLD})_{\mathrm{in}}$ has smoother properties and is more important than $(\mathsf{RLD})_{\mathrm{out}}$. (However see also the exercize below for $(\mathsf{RLD})_{\mathrm{in}}$ not preserving identities.)

Proposition 37 $(\mathsf{RLD})_{\mathrm{out}} \uparrow^{\mathsf{FCD}(A;B)} f = \uparrow^{\mathsf{RLD}(A;B)} f$ *for every small sets A, B and binary relation $f \subseteq A \times B$.*

Proof $(\mathsf{RLD})_{\mathrm{out}} \uparrow^{\mathsf{FCD}(A;B)} f = \bigcap \left\langle \uparrow^{\mathsf{RLD}(A;B)} \right\rangle \mathrm{up} \uparrow^{\mathsf{FCD}(A;B)} f = \uparrow^{\mathsf{RLD}(A;B)} \min \mathrm{up} \uparrow^{\mathsf{FCD}(A;B)} f = \uparrow^{\mathsf{RLD}(A;B)} f$. □

Surprisingly a funcoid is greater inward than outward:

Theorem 65 $(\mathsf{RLD})_{\mathrm{out}} f \subseteq (\mathsf{RLD})_{\mathrm{in}} f$ *for every funcoid f.*

Proof We need to prove

$$\bigcap \left\langle \uparrow^{\mathsf{RLD}(\mathrm{Src}\, f; \mathrm{Dst}\, f)} \right\rangle \mathrm{up}\, f \subseteq \bigcup \left\{ \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B} \quad | \quad \mathcal{A}, \mathcal{B} \in \mathfrak{F}, \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \subseteq f \right\}.$$

Let

$$K \in \mathrm{up} \bigcup \left\{ \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B} \quad | \quad \mathcal{A}, \mathcal{B} \in \mathfrak{F}, \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \subseteq f \right\}.$$

Then

$$\begin{aligned} K &= \uparrow^{\mathsf{RLD}(\mathrm{Src}\, f; \mathrm{Dst}\, f)} \bigcup \left\{ X_{\mathcal{A}} \times Y_{\mathcal{B}} \quad | \quad \mathcal{A}, \mathcal{B} \in \mathfrak{F}, \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \subseteq f \right\} \\ &= \bigcup \left\{ \uparrow^{\mathsf{RLD}(\mathrm{Src}\, f; \mathrm{Dst}\, f)} (X_{\mathcal{A}} \times Y_{\mathcal{B}}) \quad | \quad \mathcal{A}, \mathcal{B} \in \mathfrak{F}, \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \subseteq f \right\} \\ &\supseteq f \end{aligned}$$

where $X_{\mathcal{A}} \in \mathrm{up}\, \mathcal{A}$, $Y_{\mathcal{B}} \in \mathrm{up}\, \mathcal{B}$. So $K \in \mathrm{up}\, f$; $K \in \mathrm{up} \bigcap \left\langle \uparrow^{\mathsf{RLD}(\mathrm{Src}\, f; \mathrm{Dst}\, f)} \right\rangle \mathrm{up}\, f$. □

Theorem 66 $(\mathsf{FCD})(\mathsf{RLD})_{\mathrm{in}} f = f$ *for every funcoid f.*