Proof Let a be an atomic filter object on $\operatorname{Src} f$.

$\langle (\mathsf{FCD})f\rangle a = \bigcap \left\{ \left\langle \uparrow^{\mathsf{FCD}(\operatorname{Src} f;\operatorname{Dst} f)} F \right\rangle a \mid F \in \operatorname{up} f \right\}$ by the definition of (FCD).

$\left\langle \bigcap \left\langle \uparrow^{\mathsf{FCD}(\operatorname{Src} f;\operatorname{Dst} f)} \right\rangle \operatorname{up} f \right\rangle a = \bigcap \left\{ \left\langle \uparrow^{\mathsf{FCD}(\operatorname{Src} f;\operatorname{Dst} f)} F \right\rangle a \mid F \in \operatorname{up} f \right\}$ by the theorem 17.

So $\langle (\mathsf{FCD})f\rangle a = \left\langle \bigcap \left\langle \uparrow^{\mathsf{FCD}(\operatorname{Src} f;\operatorname{Dst} f)} \right\rangle \operatorname{up} f \right\rangle a$ for every a. □

Lemma 4 *For every two filter bases S and T of binary relations on $U \times V$ for some small sets U, V and every set $A \subseteq U$*

$$\bigcap \left\langle \uparrow^{\mathsf{RLD}(U;V)} \right\rangle S = \bigcap \left\langle \uparrow^{\mathsf{RLD}(U;V)} \right\rangle T \Rightarrow \bigcap \left\{ \uparrow^V \langle F\rangle A \mid F \in S \right\} = \bigcap \left\{ \uparrow^V \langle G\rangle A \mid G \in T \right\}.$$

Proof Let $\bigcap \left\langle \uparrow^{\mathsf{RLD}(U;V)} \right\rangle S = \bigcap \left\langle \uparrow^{\mathsf{RLD}(U;V)} \right\rangle T$.

First let prove that $\{\langle F\rangle A \mid F \in S\}$ is a filter base. Let $X, Y \in \{\langle F\rangle A \mid F \in S\}$. Then $X = \langle F_X\rangle A$ and $Y = \langle F_Y\rangle A$ for some $F_X, F_Y \in S$. Because S is a filter base, we have $S \ni F_Z \subseteq F_X \cap F_Y$. So $\langle F_Z\rangle A \subseteq X \cap Y$ and $\langle F_Z\rangle A \in \{\langle F\rangle A \mid F \in S\}$. So $\{\langle F\rangle A \mid F \in S\}$ is a filter base.

Suppose $X \in \operatorname{up} \bigcap \left\{ \uparrow^V \langle F\rangle A \mid F \in S \right\}$. Then exists $X' \in \{\langle F\rangle A \mid F \in S\}$ where $X \supseteq X'$ because $\{\langle F\rangle A \mid F \in S\}$ is a filter base. That is $X' = \langle F\rangle A$ for some $F \in S$. There exists $G \in T$ such that $G \subseteq F$ because T is a filter base. Let $Y' = \langle G\rangle A$. We have $Y' \subseteq X' \subseteq X$; $Y' \in \{\langle G\rangle A \mid G \in T\}$; $Y' \in \operatorname{up} \bigcap \left\{ \uparrow^V \langle G\rangle A \mid G \in T \right\}$; $X \in \operatorname{up} \bigcap \left\{ \uparrow^V \langle G\rangle A \mid G \in T \right\}$. The reverse is symmetric. □

Lemma 5 $\{G \circ F \mid F \in \operatorname{up} f, G \in \operatorname{up} g\}$ *is a filter base for every reloids f and g.*

Proof Let denote $D = \{G \circ F \mid F \in \operatorname{up} f, G \in \operatorname{up} g\}$. Let $A \in D \wedge B \in D$. Then $A = G_A \circ F_A \wedge B = G_B \circ F_B$ for some $F_A, F_B \in \operatorname{up} f$ and $G_A, G_B \in \operatorname{up} g$. So $A \cap B \supseteq (G_A \cap G_B) \circ (F_A \cap F_B) \in D$ because $F_A \cap F_B \in \operatorname{up} f$ and $G_A \cap G_B \in \operatorname{up} g$. □

Theorem 63 $(\mathsf{FCD})(g \circ f) = ((\mathsf{FCD})g) \circ ((\mathsf{FCD})f)$ *for every composable reloids f and g.*

Proof

$$\begin{aligned} \langle (\mathsf{FCD})(g \circ f)\rangle^* X &= \bigcap \left\{ \uparrow^{\operatorname{Dst} g} \langle H\rangle X \mid H \in \operatorname{up}(g \circ f) \right\} \\ &= \bigcap \left\{ \uparrow^{\operatorname{Dst} g} \langle H\rangle X \mid H \in \operatorname{up} \bigcap \left\{ \uparrow^{\mathsf{RLD}(\operatorname{Src} f;\operatorname{Dst} g)} (G \circ F) \mid F \in \operatorname{up} f, G \in \operatorname{up} g \right\} \right\}. \end{aligned}$$

Obviously

$$\bigcap \left\{ \uparrow^{\mathsf{RLD}(\operatorname{Src} f;\operatorname{Dst} g)} (G \circ F) \mid F \in \operatorname{up} f, G \in \operatorname{up} g \right\} = \bigcap \left\langle \uparrow^{\mathsf{RLD}(\operatorname{Src} f;\operatorname{Dst} g)} \right\rangle \operatorname{up} \bigcap \left\{ \uparrow^{\mathsf{RLD}(\operatorname{Src} f;\operatorname{Dst} g)} (G \circ F) \mid F \in \operatorname{up} f, G \in \operatorname{up} g \right\};$$