$0^{\mathsf{RLD}(A;C)} \Leftrightarrow \forall F \in \operatorname{up} f, G \in \operatorname{up} g, H \in \operatorname{up} h : G \circ F \not\asymp H$ (used properties of generalized filter bases).

Similarly $g \not\asymp h \circ f^{-1} \Leftrightarrow \forall F \in \operatorname{up} f, G \in \operatorname{up} g, H \in \operatorname{up} h : G \not\asymp H \circ F^{-1}$.

Thus $g \circ f \not\asymp h \Leftrightarrow g \not\asymp h \circ f^{-1}$ because $G \circ F \not\asymp H \Leftrightarrow G \not\asymp H \circ F^{-1}$ by the proposition 1. □

4.2 Direct product of filter objects

Definition 46 ***Reloidal product*** *of filter objects $\mathcal{A}$ and $\mathcal{B}$ is defined by the formula*

$$\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B} \overset{\text{def}}{=} \bigcap \left\{ \uparrow^{\mathsf{RLD}(\operatorname{Base}(\mathcal{A});\operatorname{Base}(\mathcal{B}))} (A \times B) \quad | \quad A \in \operatorname{up} \mathcal{A}, B \in \operatorname{up} \mathcal{B} \right\}.$$

Obvious 20. $\uparrow^U A \times^{\mathsf{RLD}} \uparrow^V B = \uparrow^{\mathsf{RLD}(U;V)} (A \times B)$ for every small sets $A \subseteq U$ and $B \subseteq V$.

Theorem 46 $\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B} = \bigcup \left\{ a \times^{\mathsf{RLD}} b \quad | \quad a \in \operatorname{atoms} \mathcal{A}, b \in \operatorname{atoms} \mathcal{B} \right\}$ *for every filter objects $\mathcal{A}$, $\mathcal{B}$.*

Proof Obviously

$$\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B} \supseteq \bigcup \left\{ a \times^{\mathsf{RLD}} b \quad | \quad a \in \operatorname{atoms} \mathcal{A}, b \in \operatorname{atoms} \mathcal{B} \right\}.$$

Reversely, let

$$K \in \operatorname{up} \bigcup \left\{ a \times^{\mathsf{RLD}} b \quad | \quad a \in \operatorname{atoms} \mathcal{A}, b \in \operatorname{atoms} \mathcal{B} \right\}.$$

Then $K \in \operatorname{up}(a \times^{\mathsf{RLD}} b)$ for every $a \in \operatorname{atoms} \mathcal{A}, b \in \operatorname{atoms} \mathcal{B}$; $K \supseteq X_a \times Y_b$ for some $X_a \in \operatorname{up} a$, $Y_b \in \operatorname{up} b$; $K \supseteq \bigcup \{ X_a \times Y_b \quad | \quad a \in \operatorname{atoms} \mathcal{A}, b \in \operatorname{atoms} \mathcal{B} \} = \bigcup \{ X_a \quad | \quad a \in \operatorname{atoms} \mathcal{A} \} \times \bigcup \{ Y_b \quad | \quad b \in \operatorname{atoms} \mathcal{B} \} \supseteq A \times B$ where $A \in \operatorname{up} \mathcal{A}$, $B \in \operatorname{up} \mathcal{B}$; $K \in \operatorname{up}(\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B})$. □

Theorem 47 *If $\mathcal{A}_0, \mathcal{A}_1 \in \mathfrak{F}(A)$, $\mathcal{B}_0, \mathcal{B}_1 \in \mathfrak{F}(B)$ for some small sets A, B then*

$$(\mathcal{A}_0 \times^{\mathsf{RLD}} \mathcal{B}_0) \cap (\mathcal{A}_1 \times^{\mathsf{RLD}} \mathcal{B}_1) = (\mathcal{A}_0 \cap \mathcal{A}_1) \times^{\mathsf{RLD}} (\mathcal{B}_0 \cap \mathcal{B}_1).$$

Proof

$$\begin{aligned}
& (\mathcal{A}_0 \times^{\mathsf{RLD}} \mathcal{B}_0) \cap (\mathcal{A}_1 \times^{\mathsf{RLD}} \mathcal{B}_1) \\
= \; & \bigcap \left\{ \uparrow^{\mathsf{RLD}(A;B)} (P \cap Q) \quad | \quad P \in \operatorname{up}(\mathcal{A}_0 \times^{\mathsf{RLD}} \mathcal{B}_0), Q \in \operatorname{up}(\mathcal{A}_1 \times^{\mathsf{RLD}} \mathcal{B}_1) \right\} \\
= \; & \bigcap \left\{ \uparrow^{\mathsf{RLD}(A;B)} ((A_0 \times B_0) \cap (A_1 \times B_1)) \quad | \quad A_0 \in \operatorname{up} \mathcal{A}_0, B_0 \in \operatorname{up} \mathcal{B}_0, A_1 \in \operatorname{up} \mathcal{A}_1, B_1 \in \operatorname{up} \mathcal{B}_1 \right\} \\
= \; & \bigcap \left\{ \uparrow^{\mathsf{RLD}(A;B)} ((A_0 \cap A_1) \times (B_0 \cap B_1)) \quad | \quad A_0 \in \operatorname{up} \mathcal{A}_0, B_0 \in \operatorname{up} \mathcal{B}_0, A_1 \in \operatorname{up} \mathcal{A}_1, B_1 \in \operatorname{up} \mathcal{B}_1 \right\} \\
= \; & \bigcap \left\{ \uparrow^{\mathsf{RLD}(A;B)} (K \times L) \quad | \quad K \in \operatorname{up}(\mathcal{A}_0 \cap \mathcal{A}_1), L \in \operatorname{up}(\mathcal{B}_0 \cap \mathcal{B}_1) \right\} \\
= \; & (\mathcal{A}_0 \cap \mathcal{A}_1) \times^{\mathsf{RLD}} (\mathcal{B}_0 \cap \mathcal{B}_1).
\end{aligned}$$

□